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BUCKLING OF A SPHERICAL SANDWICH SHELL
UNDER UNIFORM EXTERNAL PRESSURE

A THESIS

Presented to

The Faculty of the Graduate Division

by

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In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

in the School of Engineering Mechanics

Georgia Institute of Technology

May, 1966

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Approved:

Chairman

Date approved by Chairman: 5/16/66

This thesis is dedicated to my wife.

ACKNOWLEDGMENTS

I wish to express especial thanks to my thesis advisor, Dr. M. E. Raville, for his expert guidance, patience, and interest in the preparation of this thesis. I also wish to thank the members of my thesis committee, Dr. C. E. Stoneking, Dr. J. T. S. Wang, Professor F. M. Hill, Dr. J. M. Osborn and Dr. I. E. Perlin, for their helpful suggestions. In addition I wish to acknowledge the valuable help of Mr. H. O. Burnside in the preparation of the computer programs.

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INTRODUCTION

1. Statement of the Problem

This thesis is devoted to the analysis of the linear elastic stability of a complete spherical sandwich shell subjected to a uniform external normal pressure. A sandwich shell is a composite structure composed of two facings which behave as thin shells or membranes and a core which separates and is bonded to the facings. It is assumed that the facings transmit all of the loads occurring in the surface of the shell and that the core transmits only normal and shear stresses in the radial direction. Since the core material is usually much less dense than the facings, the sandwich shell offers a much higher stiffness to weight ratio than any comparable monocoque shell. The assumption that the core is weak in the direction of the surface of the shell is the same as assuming that the core is in a state of antiplane stress; that is, the stresses in the direction of the surface of the shell are all zero. Two types of instability or buckling behavior may occur in the sandwich shell, global and local (ripple-type) buckling. In the global buckling mode, the top and bottom facings deform in the same direction and shape. In the local buckling mode the top and bottom facings deform in the same shape but in opposite directions. Both types of buckling are analyzed.

The only previous investigation of this problem appears to be that of Yao [5]*. His analysis is discussed in detail in Appendix B.

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*Numbers in brackets refer to references at the end of the Introduction.

2. Models for the Buckling Problem

Two separate analyses of the buckling problem are made. The first, which appears in Chapter I, employs Reissner's sandwich shell theory [2]. This is the same theory used in Yao's analysis. This theory yields global buckling pressures but does not include local buckling effects. A quadratic equation is shown to govern the buckling, and lengthy computer methods are not required.

The second analysis, which appears in Chapter II, is a more general formulation of the problem. It is assumed that the sandwich shell facings are governed by the Kirchhoff-Love theory including linear change-of-curvature terms. The core is assumed to be in a state of antiplane stress and without further approximations is treated as an elastic continuum. The conditions of continuity of displacements and stresses at the shell interfaces are enforced. This model yields both global and local buckling pressures. A simple solution for the global buckling is found on the basis of the commonly used assumption of a core with infinite modulus of elasticity E_z in the radial direction (but finite modulus of rigidity G_z). An analysis of ripple buckling is carried out using the general formulation and a method is presented for distinguishing between the global and local buckling pressures.

Both analyses involve linear equations only; a nonlinear formulation is not the subject of this thesis.

3. Mathematical Assumptions

In the analyses of Chapters I and II the displacement functions of the sandwich shell are represented by infinite series of the eigenfunctions of the differential equations (Legendre polynomials). These infinite series are differentiated term-by-term as many times as needed. Since buckling actually occurs in a single eigenfunction, no proofs for manipulations of the infinite series are given. All other calculations follow rigorously from these manipulations.

4. Symmetric Deflections

Only the symmetric deformations of a spherical shell with respect to an arbitrary diameter are considered. The sufficiency of such an analysis for the linearized shell buckling equations was shown by Van der Neut [3] and a plausibility argument is given by Flügge [1]. A somewhat different approach yielding the same result is given by Vlasov [4].

5. Results and Conclusions

The results and conclusions of Chapters I and II are presented in Chapter III. Comparisons between the various theories are made.

Literature Cited in the Introduction

1. Flügge, W., Stresses in Shells, Second Printing, Springer-Verlag, Berlin, 1962, pp. 472-478.

2. Reissner, E., "Small Bending and Stretching of Sandwich-Type Shells," NACA Technical Note 1832, March 1949.

3. Van der Neut, A.,Dissertation, Delft, 1932.

4. Vlasov, V. Z., General Theory of Shells and Its Applications in Engineering, NASA Technical Translation F-99, April 1964, pp. 527-529.

5. Yao, J. C., "Buckling of Sandwich Sphere under Normal Pressure," Journal of the Aeronautical Sciences, March 1962, pp. 264-268, p. 305.

CHAPTER I

THE SIMPLIFIED LINEAR THEORY

1. Summary

The elastic stability of a complete sandwich sphere is investigated. Reissner's small deflection sandwich shell theory [5] is used. The facings are treated as membranes of equal thickness and the core is assumed to be in a state of antiplane stress. With this theory, only buckling of the shell as a whole may be considered. The reduction to the classical buckling pressure for a complete monocoque sphere is made. In analogy with the usual monocoque analysis, a quadratic equation for buckling modes of nonzero wavelength is obtained. In addition, a third possible buckling mode, which occurs for zero buckling wavelength, is established. Thus, a simple "closed form" solution to the problem is achieved. A simple computer program is given to evaluate the numerous parameters involved.

2. Notation

a	radius of middle surface of sandwich sphere
h	core-layer thickness
t	face-layer thickness
φ,θ	surface coordinates on spherical shell
$N_{\varphi u}, N_{\theta u}$	direct stress resultants in upper face membrane
$N_{\varphi \ell}, N_{\theta \ell}$	direct stress resultants in lower face membrane
Q_φ, Q_θ	transverse shear stress resultants in core

N_φ, N_θ	direct stress resultants parallel to middle surface for composite shell
M_φ, M_θ	stress couples for composite shell
q_u, q_ℓ	normal components of external load intensity on upper and lower face membranes
q	normal component of external load intensity for composite shell
s	external load intensity defined in text
E_f, G_f, ν	elastic moduli for face-layer material
E_c, G_c	elastic moduli in radial direction for core-layer material
u, v	tangential components of displacement of middle surface of core
w	normal component of displacement of middle surface of core
β	change of slope of normal to middle surface of core

$C^* = 2tE_f$

$D^* = \frac{t}{2}(h+t)^2 E_f$

$C = C^*/(1-\nu^2)$ tensile stiffness factor

$D = D^*/(1-\nu^2)$ bending stiffness factor

Parameters arising in the analysis

$\lambda = (h+t)tE_f/(2a^2E_c)$

$J = 2tE_f/\{a[1+2\lambda(1+\nu)/3 - \nu^2]\}$

$K = t(h+t)^2E_f/\{2a[1+2\lambda(1+\nu) - \nu^2]\}$

$c_1 = J(1+\lambda/3)$

$c_2 = J(\nu - \lambda/3)$

$c_3 = J(1+\nu)$

$c_4 = -J(1+\nu)(h+t)/(24E_ca)$

$d_1 = K(1+\lambda)$

$d_2 = K(\nu - \lambda)$

$d_3 = -K(1 + \nu)t(h + t)E_f/[4a^3(1 + 2\lambda - \nu)E_c^2]$

$g = (h + t)G_c$

$\lambda_n = n(n + 1)$

The sandwich shell configuration and sign conventions are shown in Figures 1 and 2.

3. Introduction

For the classical linear buckling analysis of a complete monocoque sphere subjected to uniform external pressure, one uses the linear shell theory of Love which accounts for the change between undeformed and deformed geometry [1, 3, 7]. In this approach, the shell stresses are divided into two parts: a uniform prebuckling stress and an incremental buckling stress. Numerous approximations are made before the classical buckling pressure $q_{cr} = \dfrac{2Eh^2}{a^2\sqrt{3(1-\nu)^2}}$ is derived. All of these approximations are, however, consistent with the approximations introduced by the underlying Kirchhoff-Love assumptions of the shell theory employed [2,6].

An alternative approach (seemingly less refined), would be as follows:

A. Use Love's simplified shell theory which does not account for curvature changes due to deformation [8].

B. Split the problem into two separate problems, that of the uniform prebuckling state and that of the buckling state. Use the uniform external pressure as the loading for the prebuckling problem, and

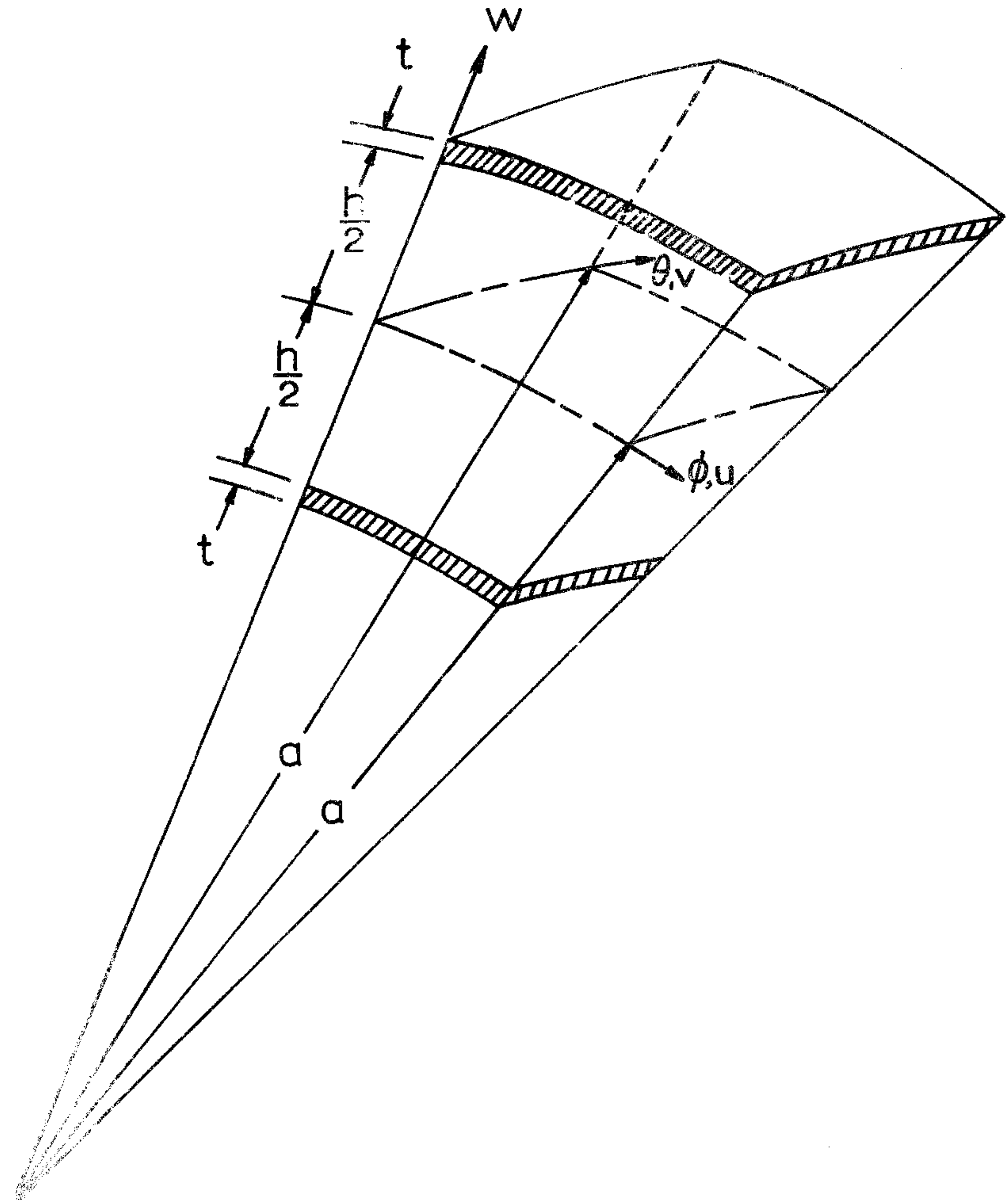

Figure 1. Coordinate Directions and Dimensions for Sandwich Sphere.

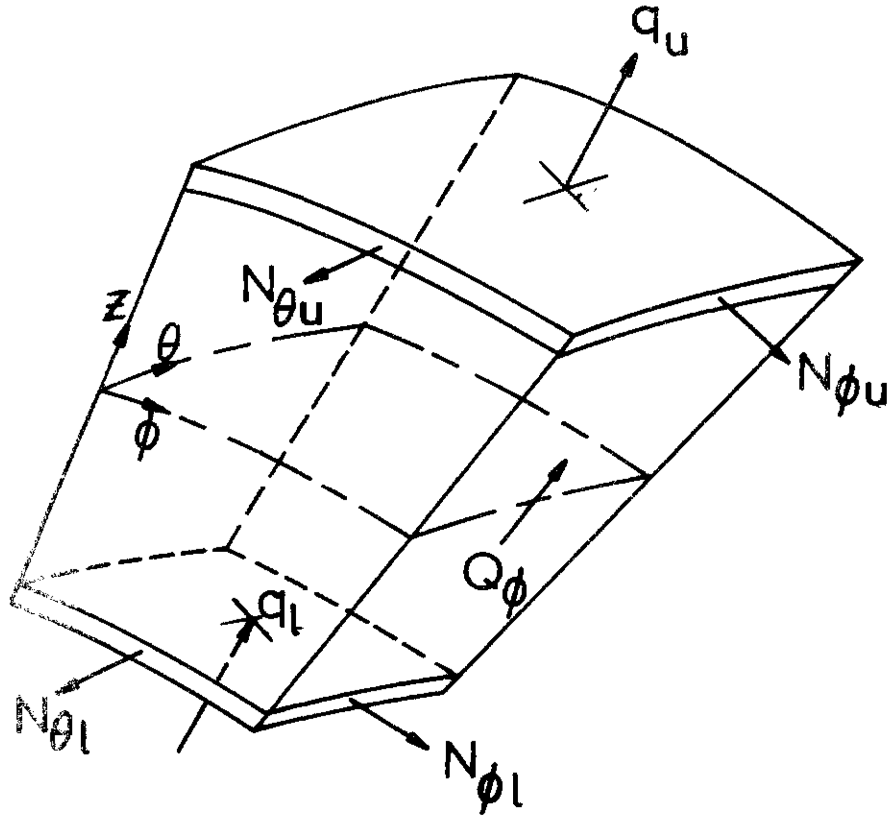

Figure 2. Stress Resultants and External Loads for Sandwich Sphere.

use the radial components of the prebuckling stress, due to curvature changes, as the loading for the buckling problem. Thus, in this approach, the deformations are decomposed into two parts: a prebuckling deformation and a buckling deformation.

It can be shown by a direct analysis(*) that this approach to the buckling problem yields the <u>same</u> classical critical buckling pressure as the customary analysis. Hence, within the scope of the approximations usually made in linear stability studies of complete spherical shells, one may use the simple shell theory(**). For the buckling problem, however, a radial "loading" $Z = -N_o \left(\frac{d}{ad\varphi} + \frac{\cot\varphi}{a}\right)\left(\frac{dw}{ad\varphi} + \frac{u}{a}\right)$ is employed.

This method will be used to calculate the classical buckling pressure for a complete sandwich sphere; the corresponding sandwich shell theory is that of Reissner [5]. It is analogous to Love's simplified theory for monocoque shells. The notation is Reissner's [5].

The only previous analysis of this problem was given by Yao [9], in 1962. He used Reissner's theory together with the Mushtari-Vlasov simplification of the theory of shells [4]. Further approximations were made in the stress-displacement relations of Reissner. The buckling was assumed to take place as a small dimple (an experimental fact in accord with nonlinear theory, but <u>not</u> predicted by the linear theory), and the

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(*)See Appendix A.

(**)Ref. [8], p. 534, eq. (312). In formulating these equations, Timoshenko says, "Assuming that the membrane forces N_θ and N_φ do not approach their critical values, we neglect the change of curvature in deriving the equations of equilibrium......." The analysis in the appendix shows that these restrictions need not be made, at least for the spherical shell, provided that a "loading" for the buckling problem is used which accounts for curvature changes.

dimple was analyzed using shallow shell theory(*).

The present analysis differs from Yao's [9] in that no approximations in the equilibrium equations of Reissner are made, and spherical coordinates are used. The resulting derivation then follows directly from the linear theory. In addition, the search for a minimum buckling pressure in the present analysis leads (without additional approximation) to a quadratic equation. Thus, a computer minimization procedure is not required.

4. Sandwich Shell Theory

The basic assumptions for the linear theory (Reissner [5]) are

1. The facings are of equal thickness and are of the same isotropic elastic material. The flexural rigidity of the facings is neglected.

2. The core is connected to the facings at their middle surfaces.

3. The core can transmit only normal stresses in the radial direction and transverse shear stresses.

4. Terms of the order of $\frac{h}{a}$ and $\frac{t}{a}$ may be neglected in comparison with unity.

The equilibrium equations for the facings and core are combined to give a set of equilibrium equations for the composite shell in terms of the composite stress resultants. The principle of complementary energy is then used to derive the linear stress-displacement relations for the complete shell. In the following, only axisymmetric deformations are considered.

(*) For a critique of Yao's paper, see Appendix B.

The axisymmetric equilibrium equations for the composite spherical shell with normal loadings only are(*)

$$\frac{d}{d\varphi}(N_\varphi \sin\varphi) - N_\theta \cos\varphi + Q_\varphi \sin\varphi$$

$$\frac{d}{d\varphi}(Q_\varphi \sin\varphi) - \sin\varphi(N_\varphi + N_\theta) + aq\sin\varphi = 0$$

$$\frac{d}{d\varphi}(M_\varphi \sin\varphi) - M_\theta \cos\varphi - aQ_\varphi \sin\varphi = 0$$

The stress-displacement relations are (*)

$$(1 + \frac{\lambda}{3})N_\varphi - (\nu - \frac{\lambda}{3})N_\theta = C^*[\frac{1}{a}\frac{du}{d\varphi} + \frac{w}{a} + \frac{h+t}{12a}\frac{q}{E_c}]$$

$$(1 + \frac{\lambda}{3})N_\theta - (\nu - \frac{\lambda}{3})N_\varphi = C^*[\frac{u\cot\varphi}{a} + \frac{w}{a} + \frac{h+t}{12a}\frac{q}{E_c}]$$

$$Q_\varphi = (h+t)G_c[\beta_\varphi + \frac{1}{a}\frac{dw}{d\varphi} - \frac{u}{a}]$$

$$(1 + \lambda)M_\varphi - (\nu - \lambda)M_\theta = \frac{D^*}{a}[\frac{d\beta}{d\varphi} + \frac{s}{E_c}]$$

$$(1 + \lambda)M_\theta - (\nu - \lambda)M_\varphi = \frac{D^*}{a}[\beta\cot\varphi + \frac{s}{E_c}]$$

where

$$\lambda = (h + t)tE_f/2a^2E_c$$

The loading parameters q and s are given by (**)

(*) Ref. [5], p. 76.
(**) Ref. [5], p. 9.

$$q = (1 + \frac{h+t}{2a})^2 \, q_u + (1 - \frac{h+t}{2a})^2 \, q_\ell$$

$$s = \frac{1}{2} \left[(1 + \frac{h+t}{2a})^2 \, q_u - (1 - \frac{h+t}{2a})^2 \, q_\ell \right]$$

5. Prebuckling Problem

Prior to buckling, the stress resultants and stress couples of the sandwich sphere under uniform external pressure ($q_u = -p$) are found from the equilibrium equations (4.1) to be

$$\overline{N}_\varphi = \overline{N}_\theta = N_o = -\frac{1}{2} qa = \frac{-pa}{2} (1 + \frac{h+t}{2a})^2 \tag{5.1}$$

$$\overline{M}_\varphi = \overline{M}_\theta = M_o = \frac{-pt(h+t)^2 \, [1 + \frac{h+t}{2a})]^2}{4a(1 + 2\lambda - \nu)} \cdot \frac{E_f}{E_c}$$

The stress resultants in the separate facings are given by

$$\overline{N}_{\varphi u} = \overline{N}_{\theta u} = [\frac{1}{2} N_o + \frac{M_o}{h+t}] \,/\, [1 + \frac{h+t}{2a}] \tag{5.2}$$

$$\overline{N}_\varphi = \overline{N}_\theta = [\frac{1}{2} N_o - \frac{M_o}{h+t}] / [1 - \frac{h+t}{2a}]$$

6. Buckling Problem

Loadings

The radial components of the prebuckling stress resultants in the separate facings are

$$q_u = \overline{N}_{\varphi u} [\frac{1}{a} \frac{d}{d\varphi} + \frac{\cot\varphi}{a}][\frac{1}{a} \frac{dw}{d\varphi} - \frac{u}{a}] \,/\, [1 + \frac{h+t}{2a}]$$

$$q_{\ell} = \overline{N_{\varphi\ell}}\left[\frac{1}{a}\frac{d}{d\varphi} + \frac{\cot\varphi}{a}\right]\left[\frac{1}{a}\frac{dw}{d\varphi} - \frac{u}{a}\right] / \left[1 - \frac{h+t}{2a}\right]$$

Hence, the composite shell loadings used in the stability analysis are

$$q = \left(1 + \frac{h+t}{2a}\right)^2 q_u + \left(1 - \frac{h+t}{2a}\right)^2 q_{\ell}$$

$$= \left[\frac{N_o}{2} + \frac{M_o}{h+t} + \frac{N_o}{2} - \frac{M_o}{h+t}\right]\left[\left(\frac{1}{a}\frac{d}{d\varphi} + \frac{\cot\varphi}{a}\right)\left(\frac{1}{a}\frac{dw}{d\varphi} - \frac{u}{a}\right)\right]$$

$$= \frac{N_o}{a^2}\left(\frac{d}{d\varphi} + \cot\varphi\right)\left(\frac{dw}{d\varphi} - u\right)$$

$$s = \frac{1}{2}\left[\left(1 + \frac{h+t}{2a}\right)^2 q_u - \left(1 - \frac{h+t}{2a}\right)^2 q_{\ell}\right]$$

$$= \frac{M_o}{h+t}\left[\left(\frac{1}{a}\frac{d}{d\varphi} + \frac{\cot\varphi}{a}\right)\left(\frac{1}{a}\frac{dw}{d\varphi} - \frac{u}{a}\right)\right]$$

Stress-Displacement Relations

Equations (4.2) are solved for the stress resultants and couples and transverse shear resultants, in terms of u, w, s and q. Substitution of the foregoing values of q and s yields the stress-displacement equations for the buckling problem:

$$Q_{\varphi} = g\left[\beta + \frac{1}{a}\frac{dw}{d\varphi} - \frac{u}{a}\right], \qquad \text{where } g = (h+t)G_c \tag{6.1}$$

$$M_{\varphi} = d_1\frac{d\beta}{d\varphi} + d_2\beta\cot\varphi + d_3 p\left(\frac{d}{d\varphi} + \cot\varphi\right)\left(\frac{dw}{d\varphi} - u\right)$$

$$M_{\theta} = d_1\beta\cot\varphi + d_2\frac{d\beta}{d\varphi} + d_3 p\left(\frac{d}{d\varphi} + \cot\varphi\right)\left(\frac{dw}{d\varphi} - u\right)$$

$$N_\varphi = c_1 \frac{du}{d\varphi} + c_2 u \cot\varphi + c_3 w + c_4 p \left(\frac{d}{d\varphi} + \cot\varphi\right)\left(\frac{dw}{d\varphi} - u\right)$$

$$N_\theta = c_1 u \cot\varphi + c_2 \frac{du}{d\varphi} + c_3 w + c_4 p \left(\frac{d}{d\varphi} + \cot\varphi\right)\left(\frac{dw}{d\varphi} - u\right)$$

The coefficients in the above equations are defined in the following manner

$$c_1 = J(1 + \lambda/3) \tag{6.2}$$

$$c_2 = J(\nu - \lambda/3)$$

$$c_3 = J(1 + \nu)$$

$$c_4 = -J(1 + \nu)(h+t)/(24E_c a)$$

$$d_1 = K(1 + \lambda)$$

$$d_2 = K(\nu - \lambda)$$

$$d_3 = -K(1 + \nu)t(h+t)E_f/(4a^3(1 + 2\lambda - \nu)E_c^2)$$

$$= -(1 + \nu)\lambda K/[2a(1 + 2\lambda - \nu)E_c]$$

where

$$J = 2tE_f/a[1 + \frac{2\lambda}{3}(1 + \nu) - \nu^2] = C^*/a[1 + \frac{2\lambda}{3}(1 + \nu) - \nu^2]$$

$$K = t(h+t)^2 E_f/2a[1 + 2\lambda(1 + \nu) - \nu^2] = D^*/a[1 + \frac{2\lambda}{3}(1 + \nu) - \nu^2]$$

These relations, which are analogous to the Hooke's law relations for a monocoque sphere, involve the prebuckling stress resultant and moment for the composite shell in the terms containing d_3 and c_4, as well as the displacements. This is not the case for monocoque-spheres. This feature, as well as the presence of an explicit formula for the transverse shear resultant Q_φ, distinguishes sandwich spheres from their monocoque counterparts.

Equilibrium Equations

The equilibrium equations (4.1) with the buckling loadings q and s found above (4.3), become

$$\frac{d}{d\varphi}(N_\varphi) + (N_\varphi - N_\theta)\cot\varphi + Q_\varphi = 0 \tag{6.3}$$

$$\frac{d}{d\varphi}(M_\varphi) + (M_\varphi - M_\theta)\cot\varphi - aQ_\varphi = 0$$

$$\frac{d}{d\varphi}(Q_\varphi) + Q_\varphi \cot\varphi - (N_\varphi + N_\theta) + \frac{N_o}{a}\left(\frac{d}{d\varphi} + \cot\varphi\right)\left(\frac{dw}{d\varphi} - u\right) = 0$$

It should be noted here that the shear resultant Q_φ and moments M_φ and M_θ have opposite signs to those chosen by Timoshenko [8] and Flugge [1] for their analogous monocoque shell theory. If the sign conventions are adjusted, equations (6.3) reduce to those of Timoshenko and Flugge.

Displacement Equations

Substituting the stress-displacement equations (6.1) into the equilibrium equations (6.3) and rearranging, the governing equations become

$$c_1 \left(\frac{d^2}{d\varphi^2} + \cot\varphi \frac{d}{d\varphi} - \cot^2\varphi \right) u - c_2 u + c_3 \frac{dw}{d\varphi} + g\beta \tag{6.4}$$

$$+ c_4 p \left(\frac{d^2}{d\varphi^2} + \cot\varphi \frac{d}{d\varphi} - \cot^2\varphi - 1 + \frac{g}{ac_4 p} \right) \left(\frac{dw}{d\varphi} - u \right) = 0$$

$$d_1 \left(\frac{d^2}{d\varphi^2} + \cot\varphi \frac{d}{d\varphi} - \cot^2\varphi \right) \beta - d_2 \beta \tag{6.5}$$

$$+ d_3 p \left(\frac{d^2}{d\varphi^2} + \cot\varphi \frac{d}{d\varphi} - \cot^2\varphi - 1 \right) \left(\frac{dw}{d\varphi} - u \right) - ag\beta$$

$$-g \frac{dw}{d\varphi} + gu = 0$$

$$g \left(\frac{d}{d\varphi} + \cot\varphi \right) \beta + \frac{g}{a} \left(\frac{d^2}{d\varphi^2} + \cot\varphi \frac{d}{d\varphi} \right) w \tag{6.6}$$

$$- \frac{g}{a} \left(\frac{d}{d\varphi} + \cot\varphi \right) u - (c_1 + c_2) \left(\frac{d}{d\varphi} + \cot\varphi \right) u - 2c_3 w$$

$$-2c_4 p \left(\frac{d}{d\varphi} + \cot\varphi \right) \left(\frac{dw}{d\varphi} - u \right) + \frac{N_o}{a} \left(\frac{d}{d\varphi} + \cot\varphi \right) \left(\frac{dw}{d\varphi} - u \right) = 0$$

Suppose $u = \frac{d\psi}{d\varphi}$ and $\beta = \frac{d\pi}{d\varphi}$.

Define the operator H by $H(\cdot) = \frac{d^2(\cdot)}{d\varphi^2} + \cot\varphi \frac{d(\cdot)}{d\varphi}$. Then

$$\frac{d}{d\varphi} H(\cdot) = \frac{d^3(\cdot)}{d\varphi^3} + \cot\varphi \frac{d^2(\cdot)}{d\varphi^2} - \cot^2\varphi \frac{d(\cdot)}{d\varphi} - \frac{d(\cdot)}{d\varphi} .$$

In terms of w, ψ, and π, and the operator H, equations (6.4), (6.5) and (6.6) become, respectively,

$$\frac{d}{d\varphi}\left\{(c_1 - c_4p)H(\psi) + (c_1 - c_2 - \frac{g}{a}) + c_4pH(w)\right. \tag{6.7}$$

$$\left. + (c_3 + \frac{g}{a})w + g\pi\right\} = 0$$

$$\frac{d}{d\varphi}\left\{d_1H(\pi) + d_3pH(w) - d_3pH(\psi) + (d_1 - d_2 - ag)\pi\right. \tag{6.8}$$

$$\left. + g\psi - gw\right\} = 0$$

$$gH(\pi) - (\frac{g}{a} + c_1 + c_2 - 2c_4p + \frac{N_o}{a})H(\psi) + (\frac{g}{a} - 2c_4p + \frac{N_o}{a})H(w) \tag{6.9}$$

$$- 2c_3w = 0$$

If equations (6.7) and (6.8) are integrated with respect to φ, arbitrary constants appear on the right hand side of each equation. However, the constant of integration of equation (6.7) may be added to the displacement potential ψ <u>without affecting</u> the corresponding value of u. Proceeding similarly with the rotation potential π and the constant of integration of equation (6.8), one thus obtains (having agreed to determine the potentials ψ and π each only to within an arbitrary constant)

$$(c_1 - c_4p)H(\psi) + (c_1 - c_2 - \frac{g}{a})\psi + c_4pH(w) + (c_3 + \frac{g}{a})w + g\pi = 0 \tag{6.10}$$

$$d_1H(\pi) + d_3pH(w) - d_3pH(\psi) + (d_1 - d_2 - ag)\pi + g\psi - gw = 0 \tag{6.11}$$

$$gH(\pi) - (\frac{g}{a} + c_1 + c_2 - 2c_4 p + \frac{N_o}{a})H(\psi)$$

$$+ (\frac{g}{a} - 2c_4 p + \frac{N_o}{a})H(w) - 2c_3 w = 0 \tag{6.12}$$

Now assume that

$$\psi = \sum_{n=o}^{\infty} A_n P_n(\cos\varphi)$$

$$w = \sum_{n=o}^{\infty} B_n P_n(\cos\varphi)$$

$$\pi = \sum_{n=o}^{\infty} \Pi_n P_n(\cos\varphi)$$

where $P_n(\cdot)$ is the Legendre function of order n. Note that $H(P_n) = -\lambda_n P_n$ and $H\,H(P_n) = \lambda_n^2 P_n$, where $\lambda_n = n(n+1)$. Then the system of equations (6.7) - (6.9) becomes

$$\sum_{n=o}^{\infty} [-(c_1 - c_4 p)\lambda_n A_n + (c_1 - c_2 - \frac{g}{a})A_n - c_4 p\lambda_n B_n$$

$$+ (c_3 + \frac{g}{a})B_n + g\Pi_n] \cdot P_n(\cos\varphi) = 0 \tag{6.13}$$

$$\sum_{n=o}^{\infty} [d_3 p\lambda_n A_n + gA_n - d_3 p\lambda_n B_n - gB_n - d_1\lambda_n \Pi_n$$

$$+ (d_1 - d_2 - ag)\,\Pi_n] \cdot P_n(\cos\varphi) = 0 \tag{6.14}$$

$$\sum_{n=o}^{\infty}\Big[\Big(\frac{g}{a}+c_1+c_2-2c_4p+\frac{N_o}{a}\Big)\lambda_n A_n-\Big(\frac{g}{a}-2c_4p+\frac{N_o}{a}\Big)\cdot \tag{6.15}$$

$$\cdot\,\lambda_n B_n-2c_3B_n-g\lambda_n\Pi_n\Big]\cdot P_n(\cos\varphi)=0$$

By the completeness of the set of Legendre polynomials $P_n(\cos\varphi)$, the system (6.13) - (6.15) becomes for each $n \geq 0$,

$$\begin{bmatrix} (c_4p-c_1)\lambda_n+(c_1-c_2-\frac{g}{a}) & -c_4p\lambda_n+(c_3+\frac{g}{a}) & g \\ pd_3\lambda_n+g & -(d_3p\lambda_n+g) & -d_1\lambda_n+(d_1-d_2-ag) \\ (\frac{g}{a}+c_1+c_2-2c_4p+\frac{N_o}{a})\lambda_n & -(\frac{g}{a}-2c_4p+\frac{N_o}{a})\lambda_n-2c_3 & -g\lambda_n \end{bmatrix}\begin{bmatrix} A_n \\ B_n \\ \Pi_n \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{6.16}$$

7. Solution for the Critical Pressure

The system (6.16) of linear homogeneous equations in A_n, B_n and Π_n has a non-trivial solution if and only if the determinant of the coefficient matrix is zero. This furnishes an equation for the buckling pressure, for each n. For $n = 0$, the functions w, ψ, and Π are constants; hence, w = constant and $u = \beta = 0$. This case is disregarded for buckling, since it corresponds to a uniform prebuckling state. It will be shown below that $(\lambda_n - 2)$ is a common factor of every term in the buckling equation. Hence, one may divide the buckling equation by $(\lambda_n - 2)$ and disregard the case $n = 1$ for buckling. If the common factor is retained, <u>any</u> value of p may be inserted in the buckling equation, and the equation identically vanishes when $n = 1$. In the buckling analysis, then, only values of $n \geq 2$ will be considered.

To find the critical pressure for a given sandwich sphere, the determinant of the coefficient matrix for equations (6.16) is set equal to zero. A linear equation for p is obtained. Since buckling takes place for large values of n, λ_n may be treated as a continuous variable $[6 \leq \lambda_n < \infty]$. Then p is minimized with respect to λ_n. A quadratic equation for λ_n is obtained. Substitution of the two roots into the expression for p yields possible critical pressures. Details of the computations are given below.

Denote

$$
\begin{aligned}
A_1 &= c_4\lambda_n \\
A_2 &= d_3\lambda_n \\
A_3 &= (-2c_4 + \frac{N_o}{ap})\lambda_n \\
B_1 &= c_1 - c_2 - \frac{g}{a} - c_1\lambda_n \\
B_2 &= c_3 + \frac{g}{a} \\
B_3 &= g \\
B_4 &= -d_1\lambda_n + d_1 - d_2 - ag \\
B_5 &= (c_3 + \frac{g}{a})\lambda_n \\
B_6 &= -2c_3 - \frac{g}{a}\lambda_n \\
B_7 &= -g\lambda_n
\end{aligned}
\tag{7.1}
$$

Then the buckling determinant becomes

$$\begin{vmatrix} A_1p + B_1 & -A_1p + B_2 & g \\ A_2p + B_3 & -(A_2p + B_3) & B_4 \\ A_3p + B_5 & -A_3p + B_6 & B_7 \end{vmatrix} = 0 \tag{7.2}$$

The buckling equation is

$$Q_1^* p - Q_o^* = 0 \tag{7.3}$$

where

$$Q_o^* = B_1B_3B_7 - B_2B_4B_5 - B_3B_6g - B_3B_5g + B_2B_3B_7 + B_1B_4B_6 \tag{7.4}$$

$$Q_1^* = A_2B_1B_7 - A_1B_4B_5 + A_3B_2B_4 + A_2B_6g + A_2B_5g \tag{7.5}$$

$$- A_2B_2B_7 - A_1B_4B_6 + A_3B_1B_4$$

Using the definitions (7.1), Q_o^* and Q_1^* become

$$Q_o^* = [\lambda_n - 2] \Big\{-c_1d_1 \frac{g}{a} \lambda_n^2 + [-c_1d_1c_3 + d_1c_3c_2 + d_1c_3 \frac{g}{a} \tag{7.6}$$

$$+ c_1d_1 \frac{g}{a} - c_1d_2 \frac{g}{a}]\lambda_n + [c_3(d_1 - d_2)(c_1 - c_2)$$

$$- c_3ag(c_1 - c_2) - c_3 \frac{g}{a} (d_1 - d_2)]\Big\}$$

$$Q_1^* = [\lambda_n - 2] \Big\{[-c_1d_3g + d_1c_1(-2c_4 + \frac{N_o}{ap}) + c_3c_4d_1]\lambda_n^2 \tag{7.7}$$

$$+ [(d_1 - d_2 - ag)(2c_1c_4 - c_1 \frac{N_o}{ap} - c_3c_4) + c_3d_3g]\lambda_n\Big\}$$

Let

$$Q_o = \frac{Q_o^*}{(\lambda_n - 2)}$$

$$Q_1 = \frac{Q_1^*}{(\lambda_n - 2)}$$

Then the buckling equation is

$$Q_1 p - Q_o = 0 \tag{7.8}$$

The requirement that p be stationary is that $\frac{dp}{d\lambda_n} = 0$; or, from (7.8)

$$Q_1 \frac{dQ_o}{2\lambda_n} = Q_o \frac{dQ_1}{d\lambda_n} \tag{7.9}$$

Let

$$Q_1 = \ell_2 \lambda_n^2 + \ell_1 \lambda_n$$

$$Q_o = k_2 \lambda_n^2 + k_1 \lambda_n + k_o$$

where ℓ_1, ℓ_2, k_1 and k_2 are chosen to agree with the coefficients of λ_n displayed in equations (7.6) and (7.7). Equation (7.9) for minimization of p as a function of λ_n becomes

$$(\ell_2 k_1 - k_2 \ell_1)\lambda_n^2 + 2k_o \ell_2 \lambda_n + \ell_1 k_o = 0 \tag{7.10}$$

The two roots of equation (7.10) yield stationary values of p. However, they do not yield all possible stationary values of p. The actual formula for $\frac{dp}{d\lambda_n}$ is

$$\frac{dp}{d\lambda_n} = \frac{Q_1 \frac{dQ_o}{d\lambda_n} - Q_o \frac{dQ_1}{d\lambda_n}}{Q_1^2} \qquad (7.11)$$

Since Q_1^2 contains terms of the order of λ_n^4 and the denominator in equation (7.11) contains terms at most of order λ_n^2, we see that

$$\lim_{\lambda_n \to \infty} \frac{dp}{d\lambda_n} = 0$$

Thus, p approaches an asymptotic value

$$\lim_{\lambda_n \to \infty} p = p_{infinite} = \frac{K_2}{\ell_2} \qquad (7.12)$$

Hence, in determining p_{cr}, one must choose the minimum value among the three possible stationary values of p. A simple computer program for this method is given in Appendix C. Computations with actual shell parameters indicate that only two of the three possible stationary values of p will be real and positive.

Numerical results, comparisons and conclusions are given in Chapter III.

8. Buckling of a Sandwich Sphere with a Rigid Core

For a rigid core, $G_c \to \infty$ and $E_c \to \infty$. Then $\lambda = (h + t)tE_f/2a^2E_c \to 0$.

The equilibrium equations (6.3) for buckling are unchanged:

$$\frac{d}{d\varphi}(N_\varphi \sin\varphi) - N_\theta \cos\varphi + Q_\varphi \sin\varphi = 0 \tag{8.1}$$

$$\frac{d}{d\varphi}(M_\varphi \sin\varphi) - M_\theta \cos\varphi - Q_\varphi a \sin\varphi = 0$$

$$(N_\varphi + N_\theta) - \frac{1}{\sin\varphi}\frac{d}{d\varphi}(Q_\varphi \sin\varphi) - \frac{N_o}{a}\left(\frac{d}{d\varphi} + \cot\varphi\right)\left(\frac{dw}{d\varphi} - u\right) = 0$$

The stress displacement relations (6.1) where $\frac{1}{G_c} = \frac{1}{E_c} = \lambda = 0$, become

$$N_\varphi = \frac{2tE_f}{(1-\nu^2)a}\left[\frac{du}{d\varphi} + \nu u \cot\varphi + (1+\nu)w\right] = \frac{C}{a}\left[\frac{du}{d\varphi} + \nu u \cot\varphi + (1+\nu)w\right] \tag{8.2}$$

$$N_\theta = \frac{2tE_f}{(1-\nu^2)a}\left[u\cot\varphi + \nu\frac{du}{d\varphi} + (1+\nu)w\right] = \frac{C}{a}\left[\nu\frac{du}{d\varphi} + u\cot\varphi + (1+\nu)w\right]$$

$$M_\varphi = \frac{-t(h+t)^2E_f}{2(1-\nu^2)a^2}\left[\frac{d^2w}{d\varphi^2} - \frac{du}{d\varphi} + \nu\cot\varphi\left(\frac{dw}{d\varphi} - u\right)\right] \tag{8.3}$$

$$= \frac{D}{a^2}\left[\frac{d^2w}{d\varphi^2} - \frac{du}{d\varphi} + \nu\cot\varphi\left(\frac{dw}{d\varphi} - u\right)\right]$$

$$M_\theta = \frac{-t(h+t)^2E_f}{2(1-\nu^2)a^2}\left[\cot\varphi\left(\frac{dw}{d\varphi} - u\right) + \left(\frac{d^2w}{d\varphi^2} - \frac{du}{d\varphi}\right)\nu\right]$$

$$= \frac{D}{a^2}\left[\cot\varphi\left(\frac{dw}{d\varphi} - u\right) + \left(\frac{d^2w}{d\varphi^2} - \frac{du}{d\varphi}\right)\nu\right]$$

Equations (8.1) are exactly the same as equations (A.2) in the Appendix (for buckling of monocoque spheres). The stress-displacement relations (8.2) and (8.3) are of the same form as the Hooke's law relations (A.3) in Appendix A (for monocoque spheres); only the constant coefficients C and D of the displacement terms are different.

Proceeding exactly as in Appendix A, but using the sandwich shell expressions for C and D, the critical buckling pressure is

$$p_{cr} = \frac{C}{a} \cdot 2\sqrt{(1-v^2)\alpha} = \frac{C}{a}\, 2\sqrt{1-v^2}\sqrt{\frac{D}{a^2C}} \quad \text{where} \quad \alpha = \frac{D}{a^2C} \qquad (8.4)$$

$$p_{cr} = \frac{4t(h+t)E_f}{a^2\sqrt{1-v^2}}$$

$$p_{cr} = \frac{4}{a^2}\sqrt{C^* \cdot D}$$

But $D = \dfrac{D^*}{1-v^2} = \dfrac{t(h+t)^2 E_f}{2(1-v^2)}$ is the equivalent of the flexural rigidity

$K = \dfrac{E(2t)^3}{12(1-v^2)}$ for a monocoque sphere of thickness $2t$. As $h \to 0$,

$D \to \dfrac{t^3E_f}{2(1-v^2)} = \dfrac{(2t)^3E_f}{16(1-v^2)}$. (Note that the total thickness of a degenerate sandwich sphere with zero core-thickness h is $2t$.) The flexural rigidity K of a monocoque sphere of thickness $2t$ is $K = \dfrac{(2t)^3 E}{12(1-v^2)}$.

Thus, the bending stiffness factor D for a sandwich shell does not reduce to that of the equivalent monocoque sphere. This is due to the assumed stress distribution in the sandwich shell; the face-parallel stresses are assumed to act at the middle surface of the facings, so the stress couple resultant from this distribution differs from that of a monocoque sphere where the face parallel stresses vary linearly through the shell thickness. To reduce formula (8.4) to the case of the monocoque sphere one must replace D with K.

The critical pressure reduces, for the monocoque sphere $(h = o)$, to

$$p_{cr} = \frac{4}{a^2}\sqrt{C^* \cdot K}$$

$$= \frac{4}{a^2}\sqrt{\frac{2tE_f \cdot E_f \cdot 8t^3}{12(1-\nu^2)}}$$

$$= \frac{2E_f(2t)^2}{a^2\sqrt{3(1-\nu^2)}}\,,$$ which is the classical critical pressure.

As an alternative approach to the reduction of this analysis to that for a monocoque sphere, one may investigate directly the buckling equations (7.8) and (7.10), where the limiting values of the terms (as $G_c \to \infty$, $E_c \to \infty$) are utilized. These limiting values are calculated below.

As $G_c \to \infty$, $E_c \to \infty$

$$\lambda \to 0$$

$$J \to C^*/a(1-\nu^2) = C/a$$

$$K \to D^*/a(1-\nu^2) = D/a$$

$$c_1 \to J = C/a$$

$$c_2 \to \nu J = \nu C/a \qquad (8.5)$$

$$c_3 \to (1+\nu)J = (1+\nu)C/a$$

$$c_4 \rightarrow 0$$

$$d_1 \rightarrow K = D/a$$

$$d_2 \rightarrow \nu K - \nu D/a$$

$$c_3 \rightarrow 0$$

Equation (7.8) contains terms linear in $g = (h + t)\, G_c$. Hence one may divide each term of equation (7.8) by g; the limiting values of the terms involved become

$$Q_o \rightarrow -\frac{c_1 d_1}{a} \lambda_n^2 + \left[\frac{d_1 c_3}{a} + \frac{c_1 d_1}{a} - \frac{c_1 d_2}{a}\right] \lambda_n \qquad (8.6)$$

$$+ \left[-c_3 a(c_1 - c_2) - \frac{c_3}{a}(d_1 - d_2)\right]$$

$$Q_o \rightarrow -\frac{CD}{a^3} \lambda_n^2 + \frac{2CD}{a^3} \lambda_n - \left[(1 - \nu^2)\left(\frac{C^2}{a} + \frac{CD}{a^3}\right)\right] \qquad (8.7)$$

$$Q_1 \rightarrow -c_1 d_3 \lambda_n^2 + \left[-\frac{a}{2} c_1 + c_3 d_3\right] \lambda_n$$

$$Q_1 \rightarrow -\frac{C}{2} \lambda_n$$

Hence,

$$\ell_2 \rightarrow 0$$

$$\ell_1 \rightarrow -\frac{C}{2}$$

$$k_2 \rightarrow -\frac{CD}{a^3}$$

$$k_1 \rightarrow \frac{2CD}{a^3}$$

$$k_o \rightarrow -(1 - \nu^2)\left(\frac{C^2}{a} + \frac{CD}{a^3}\right)$$

One may remove the common factor of C from each term, since this does not affect the reduced equations (7.8) and (7.10). Thus

$$
\begin{aligned}
\ell_2 &\to 0 \\
\ell_1 &\to -1/2 \\
k_2 &\to -D/a^3 \\
k_1 &\to 2D/a^3 \\
k_0 &\to -(1-\nu^2)(C/a + D/a^3)
\end{aligned}
\tag{8.8}
$$

Equation (7.10) becomes

$$-(-\frac{D}{a^3})(-\frac{1}{2})\lambda_n^2 + (-\frac{1}{2})[-(1-\nu^2)](C/a + D/a^3) = 0$$

$$-\frac{D}{a^3}\lambda_n^2 + (1-\nu^2)(\frac{C}{a} + \frac{D}{a^3}) = 0$$

$$\lambda_n^2 = (1-\nu^2)(\frac{C}{a} + \frac{D}{a^3})\frac{a^3}{D} \tag{8.9}$$

$$= (1-\nu^2)(\frac{a^2C}{D} + 1)$$

$$\lambda_n^2 = (1-\nu^2)(\frac{1}{\alpha} + 1)$$

where $\alpha \equiv D/a^2C$ in analogy with the monocoque analysis.

Note that in this reduction

$$\frac{pa}{2C} = \frac{(1-\nu^2) + \alpha[\lambda_n^2 - 2\lambda_n + (1-\nu^2)]}{\lambda_n} \tag{8.10}$$

This result may be compared with formula (A.13) of Appendix A, which is the corresponding form for the monocoque analysis. In this chapter, $\lambda_n = n(n + 1)$, and in Appendix A, $\lambda_n = n(n + 1) - 2$. Thus it is seen that only the last term $\alpha(1 - \nu^2)$ above differs from the monocoque analysis, where the corresponding term is $\alpha(1 + \nu)^2$. This difference is due to the fact that equation (8.10) above was derived <u>without</u> making any approximations from the theory used, while α was neglected compared to one in deriving equation (A.13) of Appendix A. If in the monocoque analysis of Appendix A one retains all terms, he obtains equation (8.10) above, in complete analogy.

Using equation (8.9), equation (8.10) for p_{cr} becomes, neglecting terms of the order of $\frac{t}{a}$ compared to one,

$$p_{cr} = \frac{4\ t(h + t)E_f}{a^2\sqrt{1 - \nu^2}} \tag{8.11}$$

This is exactly equation (8.4); the remarks following equation (8.4) apply here, and the reduction to the monocoque case is complete.

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CHAPTER II

THE GENERAL LINEAR THEORY

1. Summary

A more comprehensive model than that of Chapter I is used to investigate the stability of the sandwich sphere. The facings are analyzed with Love's general shell theory, and the flexural rigidity of each facing is included. The core is assumed to be a transversely isotropic elastic continuum in a state of antiplane stress. Continuity of displacements and stresses is enforced at the interfaces between the core and facings. The interfaces are now taken to be at the inside surfaces of the facings and not at the middle surfaces. The exact solution of the boundary displacement problem for the core is obtained and used. Local buckling effects are thus included in this model; differences in facing thickness and material properties are also permitted. The reduction to the classical buckling pressure for a complete monocoque sphere is made. For the case of a rigid core (E_z is infinite), a cubic equation determines the global buckling mode. For the more complicated case which includes local buckling, a high-order polynomial equation is involved.

2. Notation

Subscripts u and ℓ denote upper-face and lower-face quantities, respectively. The superscript o denotes prebuckling quantities. Lack of a subscript u or ℓ denotes a core quantity.

Notation

a	radius of middle surface of sandwich sphere
h	core-layer thickness
t_u, t_ℓ	face-layer thickness
φ, θ	surface coordinates on spherical shell
$N_{\varphi u}, N_{\varphi \ell}, N_{\theta u}, N_{\theta \ell}$	direct stress resultants in face layers
$Q_{\varphi u}, Q_{\varphi \ell}, Q_{\theta u}, Q_{\theta \ell}$	transverse shear stress resultants in face layers
$M_{\varphi u}, M_{\varphi \ell}, M_{\theta u}, M_{\theta \ell}$	stress couples for face layers
$P_{\varphi u}, P_{\varphi \ell}$	surface loads in meridional direction on facings
$P_{zu}, P_{z\ell}$	surface loads in radial direction on facings
$R_{\theta u}, R_{\theta \ell}$	applied moment in circumferential direction on facings
E_u, E_ℓ	moduli of elasticity for facings
E_z	core modulus of elasticity in radial direction
G_z	core modulus of rigidity for transverse shear
ν_u, ν_ℓ	Poisson's ratios for facings
u_u, u_ℓ	meridional displacements of facings
u	meridional displacement of core
w_u, w_ℓ	radial displacements of facings
w	radial displacement of core
$\sigma_z, \tau_{\varphi z}$	elastic stresses in core
a_u, a_ℓ	radii of middle surfaces of facings
p	uniform external pressure
D_u, D_ℓ	flexural rigidities of facings

The sandwich shell configuration and sign conventions are shown in Figures 3 and 4.

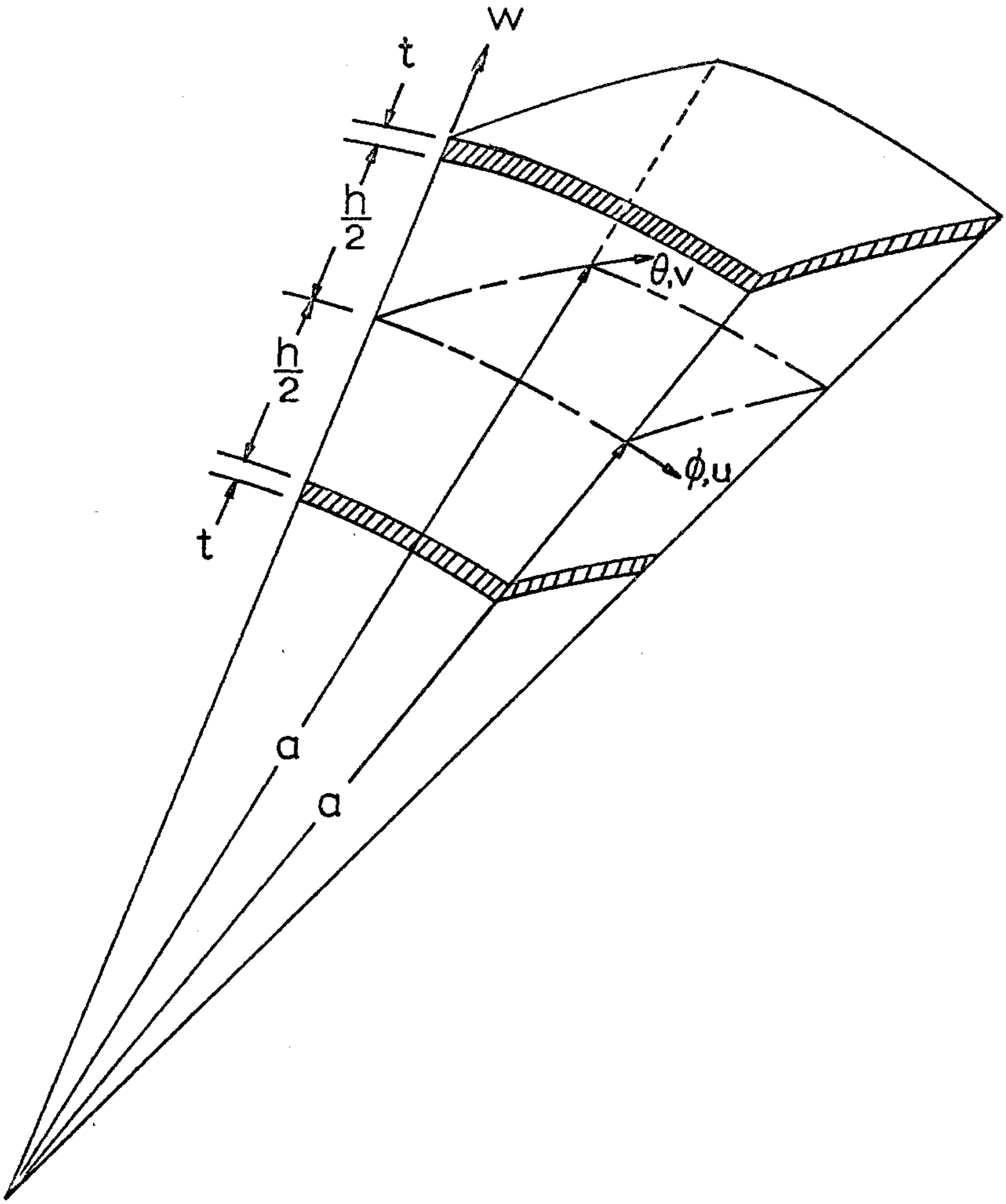

Figure 3. Coordinate Directions and Dimensions for Sandwich Sphere.

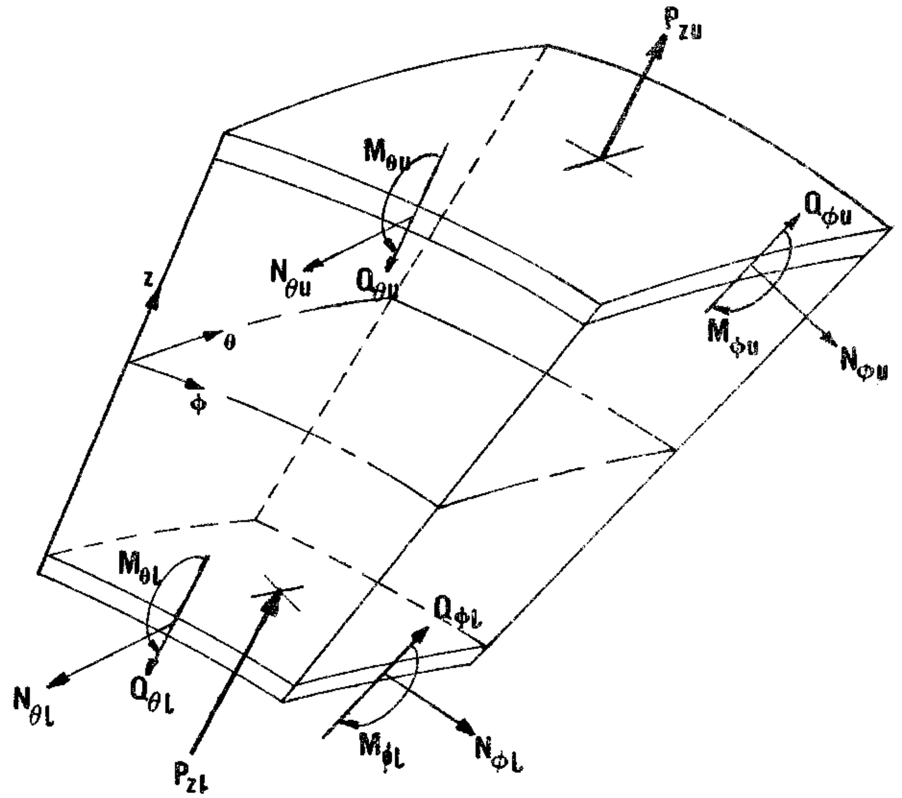

Figure 4. Stress Resultants and External Loads for Sandwich Sphere.

3. Introduction

It is desired to make a more general linear formulation of the buckling problem for a complete sandwich sphere under uniform external pressure. To that end, the facings are assumed to be isotropic elastic shells with nonzero flexural rigidities. For their analysis, one may employ Love's general shell theory including linear change-of-curvature terms and the Kirchhoff hypothesis. Further, it is assumed that the core is a transversely isotropic elastic continuum in a state of antiplane stress. No other assumptions about the core will be made.

This analysis will then include the effects of stiffness of the facings, elastic action of the core, and continuity of displacements at the sandwich shell interfaces.

4. Basic Equations for Facings

Using the coordinates shown above and modifying Love's general theory [2]* to account for the difference in his coordinate system and the one adopted above, the governing differential equations for axisymmetric deformations of a spherical shell become

$$\frac{dN_\varphi}{d\varphi} + (N_\varphi - N_\theta)\cot\varphi + Q_\varphi + N_\theta\left(\frac{u}{a} - \frac{dw}{ad\varphi}\right) + Q_\varphi\left(\frac{du}{ad\varphi} - \frac{d^2w}{ad\varphi^2}\right) + aP_\varphi = 0 \qquad (4.1)$$

$$\frac{dQ_\varphi}{d\varphi} + Q_\varphi\cot\varphi - (N_\varphi + N_\theta) - N_\varphi\frac{d}{d\varphi}\left(\frac{u}{a} - \frac{dw}{ad\varphi}\right) - N_\theta\cot\varphi\left(\frac{u}{a} - \frac{dw}{ad\varphi}\right) + aP_z = 0 \qquad (4.2)$$

*Numbers in brackets refer to references at the end of the chapter.

$$\frac{dM_\varphi}{d\varphi} + (M_\varphi - M_\theta)\cot\varphi + M_\theta\left(\frac{u}{a} - \frac{dw}{ad\varphi}\right) - aQ_\varphi + aR_\theta = 0 \qquad (4.3)$$

Here, P_φ, P_z and R_θ are the resultant surface loads in the φ and z directions and the resultant moment in the θ direction, respectively. Their assumed positive directions are indicated in Figure 3. The customary linear stress-displacement relations for a spherical shell are

$$N_\varphi = \frac{Et}{1-\nu^2}\left[\frac{du}{ad\varphi} + \frac{w}{a} + \nu\left(\frac{u\cot\varphi}{a} + \frac{w}{a}\right)\right] \qquad (4.4)$$

$$N_\theta = \frac{Et}{1-\nu^2}\left[\frac{u\cot\varphi}{a} + \frac{w}{a} + \nu\left(\frac{dw}{ad\varphi} + \frac{w}{a}\right)\right] \qquad (4.5)$$

$$M_\varphi = \frac{D}{a^2}\left[\frac{du}{d\varphi} - \frac{d^2w}{d\varphi^2} + \nu\left(u - \frac{dw}{d\varphi}\right)\cot\varphi\right] \qquad (4.6)$$

$$M_\theta = \frac{D}{a^2}\left[\left(u - \frac{dw}{d\varphi}\right)\cot\varphi + \nu\left(\frac{dw}{d\varphi} - \frac{d^2w}{d\varphi^2}\right)\right] \qquad (4.7)$$

Here $D = \dfrac{Et^3}{12(1-\nu^2)}$ is the flexural rigidity of the shell. The relations (4.1) - (4.7) are generic ones and will be applied to the two facings, keeping in mind the differences in values of radius, shell thickness, modulus of elasticity, Poisson's ratio, etc. between upper and lower facings. It is assumed that the face-parallel surface load P_φ and its associated surface moment R_θ are due solely to the distribution of shear stresses $\tau_{\varphi z}$ on the shell surfaces; hence, for a spherical shell of radius a, and thickness t,

$$P_{\varphi} = \left[\left(1 + \frac{z}{a}\right)^2 \tau_{\varphi z} \right]_{-t/2}^{t/2}$$

$$R_{\theta} = \left[\left(1 + \frac{z}{a}\right)^2 z \, \tau_{\varphi z} \right]_{-t/2}^{t/2}$$

The normal surface load P_z is due to the distribution of normal stresses σ_z on the shell surfaces; thus for a spherical shell of radius a and thickness t,

$$P_z = \left[\left(1 + \frac{z}{a}\right)^2 \sigma_z \right]_{-t/2}^{t/2}$$

The usual external loads and moments applied to a shell are taken to act at its middle surface; thus, the expressions above for P_{φ}, R_{θ} and P_z contain the proper "correction ratios" to convert a load applied to a shell surface to an equivalent load acting at the shell middle surface. Such "correction ratios" are customarily taken as unity in the Kirchhoff-Love shell theory; however there are considerable simplifications possible by their inclusion. This will also allow all approximations from the shell theory to be made at one time, in a later section.

5. Interface Displacements

It is desired to write expressions for the displacements of the sandwich facings at the interfaces where they connect to the core. Use must be made of the assumed incompressibility of the facings in the normal direction, and of the rotations due to the Kirchhoff hypothesis. Thus, one obtains:

Upper facing. Middle surface displacements are $w_u(\varphi)$, $u_u(\varphi)$.
General displacements are

$$U_u(\varphi, z_u) = u_u(\theta) + z_u\beta_u = u_u(\varphi) + z_u\left(\frac{u_u}{a_u} - \frac{1}{a_u}\frac{dw_u}{d\varphi}\right)$$

$$W_u(\varphi, z_u) = w_u(\varphi)$$

Interface displacements are

$$U_u(\varphi, -\frac{t_u}{2}) = u_u(\varphi) - \frac{t_u}{2}\left(\frac{u_u}{a_u} - \frac{1}{a_u}\frac{dw_u}{d\varphi}\right)$$

$$U_u(\varphi, -\frac{t_u}{2}) = \left(1 - \frac{t_u}{2a_u}\right) u_u(\varphi) + \frac{t_u}{2a_u}\frac{dw_u}{d\varphi} \tag{5.1}$$

$$W_u(\varphi, -\frac{t_u}{2}) = w_u(\varphi) \tag{5.2}$$

Lower facing. Middle surface displacements are $w_\ell(\varphi)$, $u_\ell(\varphi)$.
General displacements are

$$U_\ell(\varphi, z_\ell) = u_\ell(\varphi) + z_\ell\beta_\ell = u_\ell(\varphi) + z_\ell\left(\frac{u_\ell}{a_\ell} - \frac{1}{a_\ell}\frac{dw_\ell}{d\varphi}\right)$$

$$W_\ell(\varphi, z_\ell) = w_\ell(\varphi)$$

Interface displacements are

$$U_\ell(\varphi, +\frac{t_\ell}{2}) = u_\ell(\varphi) + \frac{t_\ell}{2}\left(\frac{u_\ell}{a_\ell} - \frac{1}{a_\ell}\frac{dw_\ell}{d\varphi}\right)$$

$$U_\ell(\varphi, +\frac{t_\ell}{2}) = \left(1 + \frac{t_\ell}{2a_\ell}\right) u_\ell(\varphi) - \frac{t_\ell}{2a_\ell}\frac{dw_\ell}{2\varphi} \tag{5.3}$$

$$w_\ell(\varphi, +\frac{t_\ell}{2}) = w_\ell(\varphi) \tag{5.4}$$

Continuity of displacements of the core and the facings thus requires that

$$u(\varphi, +\frac{h}{2}) = u_u(\varphi, -\frac{t_u}{2}) = (1 - \frac{t_u}{2a_u})\, u_u(\varphi) + \frac{t_u}{2a_u}\frac{dw_u}{d\varphi} \tag{5.5}$$

$$w(\varphi, +\frac{h}{2}) = w_u(\varphi) \tag{5.6}$$

$$u(\varphi, -\frac{h}{2}) = u_\ell(\varphi, +\frac{t_\ell}{2}) = (1 + \frac{t_\ell}{2a_\ell})\, u_\ell(\varphi) - \frac{t_\ell}{2a_\ell}\frac{dw_\ell}{d\varphi} \tag{5.7}$$

$$w(\varphi, -\frac{h}{2}) = w_\ell(\varphi) \tag{5.8}$$

6. Core Analysis

It is necessary to determine the elastic stresses in the core, given that the surface displacements are those of equations (5.5) - (5.8). The assumption that the core is in a state of antiplane stress means that $\sigma_{\varphi\varphi} = \sigma_{\theta\theta} = \tau_{\varphi\theta} = 0$. This is equivalent to requiring that $E_\varphi = E_\theta = 0$ in the orthotropic core. Physically these assumptions mean that the core is weak in the surface of the shell but does strengthen the shell by separating the facings and resisting radial compression and twisting. These are reasonable assumptions for the light materials customarily used for sandwich cores. Under these conditions, the core stress equilibrium equations become [1]

$$\frac{\partial}{\partial z}\left[(1 + \frac{z}{a})^3\, \tau_{\varphi z}\right] = 0 \tag{6.1}$$

$$\frac{\partial}{\partial z}\left[\left(1+\frac{z}{a}\right)^2 \sigma_z \sin\varphi\right] + \frac{d}{a d\varphi}\left[\left(1+\frac{z}{a}\right)\tau_{\varphi z} \sin\varphi\right] = 0 \qquad (6.2)$$

The stress displacement relations (with $E_\varphi = E_\theta = 0$) are

$$\tau_{\varphi z} = G_z \gamma_{\varphi z} = G_z \left\{\frac{1}{1+\frac{z}{a}} \frac{\partial w}{a\partial\varphi} + \left(1+\frac{z}{a}\right)\frac{\partial}{\partial z}\left(\frac{u}{1+\frac{z}{a}}\right)\right\} \qquad (6.3)$$

$$\sigma_z = E_z \varepsilon_z = E_z \frac{\partial w}{\partial z} \qquad (6.4)$$

Equation (6.1) implies that

$$\tau_{\varphi z} = \frac{f(\varphi)}{\left(1+\frac{z}{a}\right)^3} \qquad (6.5)$$

for some function f of φ alone. Rearranging equation (6.2), one obtains

$$\sin\varphi \frac{\partial}{\partial z}\left[\left(1+\frac{z}{a}\right)^2 \sigma_z\right] = -\left(1+\frac{z}{a}\right) \frac{d}{a d\varphi}\left[\tau_{\varphi z} \sin\varphi\right]$$

Using equation (6.5),

$$\sin\varphi \frac{\partial}{\partial z}\left[\left(1+\frac{z}{a}\right)^2 \sigma_z\right] = -\left(1+\frac{z}{a}\right) \frac{d}{a d\varphi}\left[\frac{f(\varphi)}{\left(1+\frac{z}{a}\right)^2} \sin\varphi\right] \qquad (6.6)$$

Define $g(\varphi) = -\frac{d}{a d\varphi}\left[f(\varphi) \sin\varphi\right]$.

Then, from equation (6.6),

$$\frac{\partial}{\partial z}\left[\left(1+\frac{z}{a}\right)^2 \sigma_z\right] = \frac{g(\varphi)}{\left(1+\frac{z}{a}\right)^2 \sin\varphi} = \frac{h(\varphi)}{\left(1+\frac{z}{a}\right)^2} \qquad (6.7)$$

where $h(\varphi) = \dfrac{g(\varphi)}{\sin\varphi} = -\dfrac{1}{\sin\varphi}\dfrac{d}{a d\varphi}\left[f(\varphi)\ \sin\varphi\right]$.

Hence, from equation (6.7),

$$\left(1+\frac{z}{a}\right)^2 \sigma_z = \frac{-a h(\varphi)}{1+\frac{z}{a}} + j(\varphi)$$

for some function j of φ alone.

Applying stress-displacement relations (6.4), one obtains

$$\sigma_z = E_z \frac{\partial w}{\partial z} = -\frac{a h(\varphi)}{\left(1+\frac{z}{a}\right)^3} + \frac{j(\varphi)}{\left(1+\frac{z}{a}\right)^2} \tag{6.8}$$

$$w(\varphi, z) = \frac{a^2 h(\varphi)}{2E_z\left(1+\frac{z}{a}\right)^2} - \frac{a j(\varphi)}{E_z\left(1+\frac{z}{a}\right)} + k(\varphi) \tag{6.9}$$

for some function k of φ alone.

Stress-displacement relation (6.3), together with equation (6.5), gives

$$G_z\left\{\frac{1}{1+\frac{z}{a}}\frac{\partial w}{a\partial\varphi} + \left(1+\frac{z}{a}\right)\frac{\partial}{\partial z}\left[\frac{u}{1+\frac{z}{a}}\right]\right\} = \frac{f(\varphi)}{\left(1+\frac{z}{a}\right)^3}$$

Hence,

$$\left(1+\frac{z}{a}\right)\frac{\partial}{\partial z}\left[\frac{u}{1+\frac{z}{a}}\right] = \frac{f(\varphi)}{G_z\left(1+\frac{z}{a}\right)^3} - \frac{1}{1+\frac{z}{a}}\frac{\partial w}{a\partial\varphi}$$

or, applying equation (6.9),

$$\frac{\partial}{\partial z}\left[\frac{u}{1+\frac{z}{a}}\right] = \frac{f(\varphi)}{G_z\left(1+\frac{z}{a}\right)^4} - \frac{a h'(\varphi)}{2E_z\left(1+\frac{z}{a}\right)^4} + \frac{j'(\varphi)}{E_z\left(1+\frac{z}{a}\right)^3} - \frac{k'(\varphi)}{a\left(1+\frac{z}{a}\right)^2}$$

Thus,

$$\frac{u}{1+\frac{z}{a}} = -\frac{af(\varphi)}{3G_z(1+\frac{z}{a})^3} + \frac{a^2h'(\varphi)}{6E_z(1+\frac{z}{a})^3} - \frac{aj'(\varphi)}{2E_z(1+\frac{z}{a})^2} + \frac{R'(\varphi)}{(1+\frac{z}{a})} + \ell(\varphi)$$

for some function ℓ of φ alone.

Then,

$$u(\varphi, z) = -\frac{af(\varphi)}{3G_z(1+\frac{z}{a})^2} + \frac{a^2h'(\varphi)}{6E_z(1+\frac{z}{a})^2} - \frac{aj'(\varphi)}{2E_z(1+\frac{z}{a})} + k'(\varphi)$$

$$+ (1+\frac{z}{a})\,\ell(\varphi) \qquad (6.10)$$

The conditions (5.5) - (5.8) of continuity of displacements at the sandwich shell interfaces now furnish the four boundary conditions necessary for the solution of the core displacements:

$$w_u(\varphi) = \frac{a^2h(\varphi)}{2E_z(1+\frac{h}{2a})^2} - \frac{aj(\varphi)}{E_z(1+\frac{h}{2a})} + k(\varphi) \qquad (6.11)$$

$$w_\ell(\varphi) = \frac{a^2h(\varphi)}{2E_z(1-\frac{h}{2a})^2} - \frac{aj(\varphi)}{E_z(1-\frac{h}{2a})} + k(\varphi) \qquad (6.12)$$

$$(1-\frac{t_u}{2a_u})\,u_u(\varphi) + \frac{t_u}{2a_u}\frac{dw_u}{d\varphi} = -\frac{af(\varphi)}{3G_z(1+\frac{h}{2a})^2} + \frac{a^2h'(\varphi)}{6E_z(1+\frac{h}{2a})^2}$$

$$-\frac{aj'(\varphi)}{2E_z(1+\frac{h}{2a})} + k'(\varphi) + (1+\frac{h}{2a})\,\ell(\varphi) \qquad (6.13)$$

$$(1 + \frac{t_\ell}{2a_\ell})\, u_\ell(\varphi) - \frac{t_\ell}{2a_\ell}\frac{dw_\ell}{d\varphi} = - \frac{af(\varphi)}{3G_z(1 - \frac{h}{2a})^2} + \frac{a^2h'(\varphi)}{6E_z(1 - \frac{h}{2a})^2}$$

$$- \frac{aj'(\varphi)}{2E_z(1 - \frac{h}{2a})} + k'(\varphi) + (1 - \frac{h}{2a})\, \ell(\varphi) \tag{6.14}$$

Define the differential operator L by

$$L[\circ] = \frac{d^2(\cdot)}{d\varphi^2} + \cot\varphi \frac{d(\cdot)}{d\varphi} \tag{6.15}$$

If $P_n(\cdot)$ denotes the Legendre polynomial of order n, then the following properties of L hold:

$$L[P_n(\cos\varphi)] = -\beta_n P_n(\cos\varphi) \tag{6.16}$$

$$LL[P_n(\cos\varphi)] = \beta_n^2 P_n(\cos\varphi) \tag{6.17}$$

where $\beta_n = n(n+1)$

If another differential operator M is defined by

$$M[\circ] = \frac{d(\cdot)}{d\varphi} + \cot\varphi$$

then

$$M[\frac{d}{d\varphi}(\circ)] = L[\circ] \tag{6.18}$$

Now assume that there exist potential functions $\psi_u(\varphi)$, $\psi_\ell(\varphi)$ and $\delta(\varphi)$ such that

$$\frac{d}{d\varphi} u_u(\varphi) = \psi_u(\varphi),\ \frac{d}{d\varphi} u_\ell(\varphi) = \psi_\ell(\varphi) \text{ and } \frac{d}{d\varphi}\ell(\varphi) = \delta(\varphi) \tag{6.19}$$

Applying the operator M to boundary conditions (6.13) and (6.14) and utilizing (6.18) and (6.19) leads to

$$(1 - \frac{t_u}{2a_u})\, L\psi_u(\varphi) + \frac{t_u}{2a_u}\, Lw_u(\varphi) = - \frac{a\ Mf(\varphi)}{3G_z(1 + \frac{h}{2a})^2} + \frac{a^2\ Lh(\varphi)}{6E_z(1 + \frac{h}{2a})^2}$$

$$- \frac{a\ Lj(\varphi)}{2E_z(1 + \frac{h}{2a})} + Lk(\varphi) + (1 + \frac{h}{2a})\, L\delta(\varphi) \tag{6.20}$$

$$(1 + \frac{t_\ell}{2a_\ell})\, L\psi_\ell(\varphi) - \frac{t_\ell}{2a_\ell}\, Lw_\ell(\varphi) = - \frac{a\ Mf(\varphi)}{3G_z(1 - \frac{h}{2a})^2} + \frac{a^2\ Lh(\varphi)}{6E_z(1 - \frac{h}{2a})^2}$$

$$- \frac{a\ Lj(\varphi)}{2E_z(1 - \frac{h}{2a})} + Lk(\varphi) + (1 - \frac{h}{2a})\, L\delta(\varphi) \tag{6.21}$$

Recall that $h(\varphi)$ is defined as

$$h(\varphi) = - \frac{1}{\sin\varphi} \frac{d}{a d\varphi} [f(\varphi) \sin\varphi)]$$

Thus,

$$h(\varphi) = - \frac{1}{a \sin(\varphi)} [f'(\varphi) \sin\varphi + f(\varphi) \cos\varphi] = - \frac{1}{a}[\frac{d}{d\varphi} + \cot\varphi] f(\varphi)$$

Therefore,

$$h(\varphi) = - \frac{1}{a}\, Mf(\varphi)$$

Introducing the potential function $\eta(\varphi)$ defined by

$$\frac{d}{d\varphi}\, \eta(\varphi) = f(\varphi)$$

$$h(\varphi) = -\frac{1}{a} L\eta(\varphi)$$

Then equations (6.20) and (6.21) become

$$\left(1 - \frac{t_u}{2a_u}\right) L\psi_u(\varphi) + \frac{t_u}{2a_u} Lw_u(\varphi) = \frac{a^2 h(\varphi)}{3G_z\left(1 + \frac{h}{2a}\right)^2} + \frac{a^2 Lh(\varphi)}{6E_z\left(1 + \frac{h}{2a}\right)^2}$$

$$- \frac{aLj(\varphi)}{2E_z\left(1 + \frac{h}{2a}\right)} + Lk(\varphi) + \left(1 + \frac{h}{2a}\right) L\delta(\varphi) \qquad (6.22)$$

$$\left(1 + \frac{t_\ell}{2a_\ell}\right) L\psi_\ell(\varphi) - \frac{t_\ell}{2a_\ell} Lw_\ell(\varphi) = \frac{a^2 h(\varphi)}{3G_z\left(1 - \frac{h}{2a}\right)^2} + \frac{a^2 Lh(\varphi)}{6E_z\left(1 - \frac{h}{2a}\right)^2}$$

$$- \frac{aLj(\varphi)}{2E_z\left(1 - \frac{h}{2a}\right)} + Lk(\varphi) + \left(1 - \frac{h}{2a}\right) L\delta(\varphi) \qquad (6.23)$$

Adding equations (6.11) and (6.12),

$$w_u(\varphi) + w_\ell(\varphi) = \frac{a^2}{2E_z}\left[\frac{1}{\left(1 + \frac{h}{2a}\right)^2} + \frac{1}{\left(1 - \frac{h}{2a}\right)^2}\right] h(\varphi)$$

$$= \frac{a}{E_z}\left[\frac{1}{1 + \frac{h}{2a}} + \frac{1}{1 - \frac{h}{2a}}\right] j(\varphi) + 2k(\varphi) \qquad (6.24)$$

Subtracting equation (6.12) from (6.11) ,

$$w_u(\varphi) - w_\ell(\varphi) = \frac{a^2}{2E_z}\left[\frac{1}{\left(1 + \frac{h}{2a}\right)^2} - \frac{1}{\left(1 - \frac{h}{2a}\right)^2}\right] h(\varphi)$$

$$- \frac{a}{E_z}\left[\frac{1}{1 + \frac{h}{2a}} - \frac{1}{1 - \frac{h}{2a}}\right] j(\varphi) \qquad (6.25)$$

Clearly, equations (6.24) and (6.25) are equivalent to equations (6.11) and (6.12). Simplifying these equations,

$$\frac{w_u(\varphi) + w_\ell(\varphi)}{2} = \frac{a^2}{2E_z}\left[\frac{1 + (\frac{h}{2a})^2}{[1 - (\frac{h}{2a})^2]^2}\right] h(\varphi) - \frac{a}{E_z}\left[\frac{1}{1 - (\frac{h}{2a})^2}\right] j(\varphi)$$

$$+ k(\varphi) \qquad (6.26)$$

$$\frac{w_u(\varphi) - w_\ell(\varphi)}{h} = -\frac{a}{E_z}\left[\frac{1}{[1 - (\frac{h}{2a})^2]^2}\right] h(\varphi) + \frac{1}{E_z}\left[\frac{1}{1 - (\frac{h}{2a})^2}\right] j(\varphi) \qquad (6.27)$$

Similarly, adding equations (6.22) and (6.23), one has

$$L[\psi_u(\varphi) + \psi_\ell(\varphi)] - \frac{t_u}{2a_u} L[\psi_u(\varphi) - w_u(\varphi)] + \frac{t_\ell}{2a_\ell}[\psi_\ell(\varphi) - w_\ell(\varphi)]$$

$$= \frac{a^2}{3G_z}\left[\frac{1}{(1 + \frac{h}{2a})^2} + \frac{1}{(1 - \frac{h}{2a})^2}\right] h(\varphi) + \frac{a^2}{6E_z}\left[\frac{1}{(1 + \frac{h}{2a})^2}\right.$$

$$\left. + \frac{1}{(1 - \frac{h}{2a})^2}\right] Lh(\varphi) - \frac{a}{2E_z}\left[\frac{1}{1 + \frac{h}{2a}} + \frac{1}{1 - \frac{h}{2a}}\right] Lj(\varphi) + 2Lk(\varphi) + 2L\delta(\varphi) \qquad (6.28)$$

Subtracting equation (6.23) from (6.22),

$$L\left[\psi_u(\varphi) - \psi_\ell(\varphi)\right] - \frac{t_u}{2a_u} L\left[\psi_u(\varphi) - w_u(\varphi)\right] - \frac{t_\ell}{2a_\ell} L\left[\psi_\ell(\varphi) - w_\ell(\varphi)\right]$$

$$= \frac{a^2}{3G_z}\left[\frac{1}{(1 + \frac{h}{2a})^2} - \frac{1}{(1 - \frac{h}{2a})^2}\right] h(\varphi) + \frac{a^2}{6E_z}\left[\frac{1}{(1 + \frac{h}{2a})^2}\right.$$

$$\left. - \frac{1}{(1 - \frac{h}{2a})^2}\right] Lh(\varphi) - \frac{a}{2E_z}\left[\frac{1}{(1 + \frac{h}{2a})} - \frac{1}{(1 - \frac{h}{2a})}\right] Lj(\varphi) + \frac{h}{a} L\delta(\varphi) \qquad (6.29)$$

Clearly, equations (6.28) and (6.29) are equivalent to equations (6.22) and (6.23). Simplifying, the result is

$$L\left[\frac{\psi_u(\varphi) + \psi_\ell(\varphi)}{2}\right] - \frac{t_u}{4a_u} L[\psi_u(\varphi) - w_u(\varphi)] + \frac{t_\ell}{4a_\ell} L\left[\psi_\ell(\varphi) - w_\ell(\varphi)\right]$$

$$= \frac{a^2}{3G_z}\left[\frac{1 + (\frac{h}{2a})^2}{[1 - (\frac{h}{2a})^2]^2}\right] h(\varphi) + \frac{a^2}{6E_z}\left[\frac{1 + (\frac{h}{2a})^2}{[1 - (\frac{h}{2a})^2]^2}\right] Lh(\varphi)$$

$$- \frac{a}{2E_z}\left[\frac{1}{1 - (\frac{h}{2a})^2}\right] Lj(\varphi) + Lk(\varphi) + L\delta(\varphi) \qquad (6.30)$$

$$L\left[\frac{\psi_u(\varphi) - \psi_\ell(\varphi)}{h}\right] - \frac{t_u}{2ha_u} L[\psi_u(\varphi) - w_u(\varphi)] - \frac{t}{2ha_\ell} L[\psi_\ell(\varphi) - w_\ell(\varphi)]$$

$$= - \frac{2a}{3G_z}\left[\frac{1}{[1 - (\frac{h}{2a})^2]^2}\right] h(\varphi) - \frac{a}{3E_z}\left[\frac{1}{[1 - (\frac{h}{2a})^2]^2}\right] Lh(\varphi)$$

$$+ \frac{1}{2E_z}\left[\frac{1}{1 - (\frac{h}{2a})^2}\right] Lj(\varphi) + \frac{1}{a} L\delta(\varphi) \qquad (6.31)$$

It is necessary now to perform some formal mathematical manipulations. Rigorous analytic justification of the subsequent steps is not feasible, since they involve high order derivatives of Fourier-Legendre series of functions. Instead one may note that the final solution, substituted back into the original equations, will provide its own verification.

Recall that the set of Legendre polynomials $\{P_n(\cos \varphi)\}$ is complete with respect to square integrable functions on $[-\pi, \pi]$; and expand the functions w_u, w_ℓ, ψ_u, ψ_ℓ, η, j, k and δ in Fourier-Legendre series as follows

$$\psi_u(\varphi) = \sum_{n=o}^{\infty} a_n P_n(\cos\varphi)$$

$$\psi_\ell(\varphi) = \sum_{n=o}^{\infty} b_n P_b(\cos\varphi)$$

$$w_u(\varphi) = \sum_{n=o}^{\infty} c_n P_n(\cos\varphi)$$

$$w_\ell(\varphi) = \sum_{n=o}^{\infty} d_n P_n(\cos\varphi)$$

$$h(\varphi) = \sum_{n=o}^{\infty} h_n P_n(\cos\varphi) \tag{6.32}$$

$$j(\varphi) = \sum_{n=o}^{\infty} j_n P_n(\cos\varphi)$$

$$k(\varphi) = \sum_{n=o}^{\infty} k_n P_n(\cos\varphi)$$

$$\delta(\varphi) = \sum_{n=o}^{\infty} \delta_n P_n(\cos\varphi)$$

The core problem, then, is to determine h and j in terms of the facing middle surface displacements ψ_u, ψ_ℓ, w_u and w_ℓ.

With equations (6.32), the boundary conditions (6.30), (6.31), (6.26) and (6.27) become (in matrix form)

$$
\begin{bmatrix}
\frac{a^2}{6}\left[\frac{1+(\frac{h}{2a})^2}{[1-(\frac{h}{2a})^2]^2}\right]\left[\frac{2}{G_z}-\frac{\beta_n}{E_z}\right] & \frac{a}{2E_z}\left[\frac{1}{1-(\frac{h}{2a})^2}\right]\beta_n & -\beta_n & -\beta_n \\
-\frac{a}{3}\left[\frac{1}{[1-(\frac{h}{2a})^2]^2}\right]\left[\frac{2}{G_z}-\frac{\beta_n}{E_z}\right] & -\frac{1}{2E_z}\left[\frac{1}{1-(\frac{h}{2a})^2}\right]\beta_n & 0 & -\frac{\beta_n}{a} \\
\frac{a^2}{2E_z}\left[\frac{1+(\frac{h}{2a})^2}{[1-(\frac{h}{2a})^2]^2}\right] & -\frac{a}{E_z}\left[\frac{1}{1-(\frac{h}{2a})^2}\right] & 1 & 0 \\
\frac{a}{E_z}\left[\frac{1}{[1-(\frac{h}{2a})^2]^2}\right] & \frac{1}{E_z}\left[\frac{1}{1-(\frac{h}{2a})^2}\right] & 0 & 0
\end{bmatrix}
\begin{bmatrix} h_n \\ j_n \\ \kappa_n \\ \delta_n \end{bmatrix}
=
\begin{bmatrix}
-\frac{(a_n+b_n)}{2}\beta_n+\frac{t_u}{4a_u}(a_n-c_n)\beta_n-\frac{t_\ell}{4a_\ell}(b_n-d_n)\beta_n \\
-\frac{(a_n-b_n)}{h}\beta_n+\frac{t_u}{2ha_u}(a_n-c_n)\beta_n+\frac{t_\ell}{2ha_\ell}(b_n-d_n)\beta_n \\
\frac{c_n+d_n}{2} \\
\frac{c_n-d_n}{h}
\end{bmatrix}
\tag{5.33}
$$

An equivalent system is

$$
\begin{bmatrix}
-\frac{a^2}{3}\left[\frac{1}{[1-(\frac{h}{2a})^2]^2}\right]\left\{\left(\frac{2}{G_z}-\frac{\beta_n}{E_z}\right) + \left(\frac{1}{G_z}+\frac{\beta_n}{E_z}\right)\left(1+(\frac{h}{2a})^2\right)\right\} & 0 & 0 & 0 \\
-\frac{a^2}{3}\left[\frac{1}{[1-(\frac{h}{2a})^2]^2}\right]\left[\frac{2}{G_z}-\frac{\beta_n}{E_z}\right] & -\frac{a}{2E_z}\left[\frac{1}{1-(\frac{h}{2a})^2}\right]\beta_n & 0 & -\beta_n \\
\frac{a^2}{2E_z}\left[\frac{1+(\frac{h}{2a})^2}{[1-(\frac{h}{2a})^2]^2}\right] & -\frac{a}{E_z}\left[\frac{1}{1-(\frac{h}{2a})^2}\right] & 1 & 0 \\
-\frac{a}{E_z}\left[\frac{1}{[1-(\frac{h}{2a})^2]^2}\right] & \frac{1}{E_z}\left[\frac{1}{1-(\frac{h}{2a})^2}\right] & 0 & 0
\end{bmatrix}
\begin{bmatrix} h_n \\ j_n \\ k_n \\ \delta_n \end{bmatrix}
=
\begin{bmatrix}
\left(\frac{a_n+b_n}{2}\right)\beta_n - \frac{t_u}{4a_u}(a_n-c_n)\beta_n + \frac{t_\ell}{4a_\ell}(b_n-d_n)\beta_n - \left(\frac{c_n+d_n}{2}\right)\beta_n - \frac{a}{h}(a_n-b_n)\beta_n + \frac{t_u}{2hA_u}(a_n-c_n)\beta_n + \frac{t_\ell}{2hA_\ell}(b_n-d_n)\beta_n \\
-\frac{a}{h}(a_n-b_n)\beta_n + \frac{t_u}{2hA_u}(a_n-c_n)\beta_n + \frac{t_\ell}{2hA_\ell}(b_n-d_n)\beta_n \\
\frac{c_n+d_n}{2} \\
\frac{c_n-d_n}{h}
\end{bmatrix}
\quad (6.34)
$$

where $a_u = A_u a$ and $a_\ell = A_\ell a$.

Hence,

$$h_n = \frac{3\left[1-\left(\frac{h}{2a}\right)^2\right]^2 \beta_n}{a^2\left\{\frac{2}{G_z}-\frac{\beta_n}{E_z}+\left(\frac{1}{G_z}+\frac{\beta_n}{E_z}\right)\left[1+\left(\frac{h}{2a}\right)^2\right]\right\}}\left\{\left[\frac{t_u}{4a_u}-\frac{s}{h}-\frac{1}{2}-\frac{t_u}{2hA_u}\right]a_n\right.$$

$$-\left[\frac{t_\ell}{4a_\ell}+\frac{s}{h}+\frac{1}{2}+\frac{t_\ell}{2hA_\ell}\right]b_n+\left[\frac{1}{2}-\frac{t_u}{4a_u}+\frac{t_u}{2hA_u}\right]c_n$$

$$\left.+\left[\frac{1}{2}+\frac{t_\ell}{4a_\ell}+\frac{t_\ell}{2hA_\ell}\right]d_n\right\} \qquad (6.35)$$

Also, from (6.34),

$$\frac{j_n}{E_z\left[1-\left(\frac{h}{2a}\right)^2\right]}=\frac{c_n-d_n}{h}+\frac{a}{E_z}\frac{h_n}{\left[1-\left(\frac{h}{2a}\right)^2\right]^2}$$

$$j_n=\frac{a}{\left[1-\left(\frac{h}{2a}\right)^2\right]}h_n+\left[1-\left(\frac{h}{2a}\right)^2\right]\frac{E_z}{h}\left(c_n-d_n\right) \qquad (6.36)$$

Now, by the Hooke's law expression (6.8),

$$\sigma_z(\varphi,z)=E_z\frac{\partial w(\varphi,z)}{\partial z}=\frac{a\,h(\varphi)}{\left(1+\frac{z}{a}\right)^3}+\frac{j(\varphi)}{\left(1+\frac{z}{a}\right)^2}$$

Thus, the upper interface stress is given by

$$\sigma_{zu}(\varphi)\equiv\sigma_z\left(\varphi,+\frac{h}{2}\right)=\frac{a\,h(\varphi)}{\left(1+\frac{h}{2a}\right)^3}+\frac{j(\varphi)}{\left(1+\frac{h}{2a}\right)^2}$$

$$\sigma_{zu}(\varphi)=\frac{1}{\left(1+\frac{h}{2a}\right)^2}\sum_{n=0}^{\infty}\left(\frac{a\,h_n}{1+\frac{h}{2a}}+j_n\right)\cos n\varphi$$

using equations (6.32).

Using the solution (6.36) for j_n,

$$-\frac{ah_n}{1+\frac{h}{2a}} + j_n = -\frac{ah_n}{1+\frac{h}{2a}} + \frac{a}{[1-(\frac{h}{2a})^2]} h_n + [1-(\frac{h}{2a})^2] \frac{E_z}{h} (c_n - d_n)$$

or

$$-\frac{ah_n}{1+\frac{h}{2a}} + j_n = \frac{h}{2[1-(\frac{h}{2a})^2]} h_n + [1-(\frac{h}{2a})^2] \frac{E_z}{h} (c_n - d_n)$$

Thus,

$$\sigma_{zu}(\varphi) = \frac{1}{(1+\frac{h}{2a})^2} \sum_{n=o}^{\infty} \left\{ \frac{h}{2[1-(\frac{h}{2a})^2]} h_n + [1-(\frac{h}{2a})^2] \frac{E_z}{h} (c_n - d_n) \right\} P_n (\cos \varphi) \qquad (6.37)$$

where h_n is given by equation (6.35).

Similarly,

$$\sigma_{z\ell}(\varphi) \equiv \sigma_z(\varphi, -\frac{h}{2}) = \frac{-ah(\varphi)}{(1-\frac{h}{2a})^3} + \frac{j(\varphi)}{(1-\frac{h}{2a})^2}$$

or

$$\sigma_{z\ell}(\varphi) = \frac{1}{(1-\frac{h}{2a})^2} \sum_{n=o}^{\infty} \left[-\frac{ah_n}{1-\frac{h}{2a}} + j_n \right] P_n(\cos \varphi)$$

Again using the solution (6.36) for j_n, it is found that

$$-\frac{ah_n}{1-\frac{h}{2a}} + j_n = -\frac{ah_n}{1-\frac{h}{2a}} + \frac{a}{[1-(\frac{h}{2a})^2]} h_n + [1-(\frac{h}{2a})^2] \frac{E_z}{h} (c_n - d_n)$$

or

$$-\frac{ah_n}{1-\frac{h}{2a}} + j_n = -\frac{h}{2[1-(\frac{h}{2a})^2]} h_n + [1-(\frac{h}{2a})^2] \frac{E_z}{h} (c_n - d_n)$$

Thus,

$$\sigma_{z\ell}(\varphi) = \frac{1}{(1-\frac{h}{2a})^2} \sum_{n=o}^{\infty} \left\{ -\frac{h}{2[1-(\frac{h}{2a})^2]} h_n + [1-(\frac{h}{2a})^2] \frac{E_z}{h} (c_n - d_n) \right\} P_n(\cos\varphi) \quad (6.38)$$

The shear stress $\tau_{\varphi z}$ is given by equation (6.5) as

$$\tau_{\varphi z} = \frac{f(\varphi)}{(1+\frac{z}{a})^3}$$

In the sequel, there will be no need to know $\tau_{\varphi z}$; one needs only the potential function $T(\varphi, z)$ such that

$$\frac{\partial T(\varphi, z)}{\partial \varphi} = \tau_{\varphi z}$$

Recalling the potential function $\eta(\varphi)$ defined by

$$\frac{d\eta}{d\varphi}(\varphi) = f(\varphi),$$

then

$$\frac{\partial}{\partial \varphi} \frac{\eta(\varphi)}{(1+\frac{z}{a})^3} = \tau_{\varphi z} \,,$$

where $h(\varphi) = -\frac{1}{a} L\,\eta(\varphi)$.

Expanding $\eta(\varphi)$ as $\eta(\varphi) = \sum_{n=o}^{\infty} \eta_n P_n(\cos\varphi)$

$$h(\varphi) = \sum_{n=o}^{\infty} h_n P_n(\cos\varphi) = -\frac{1}{a} L\,\eta(\varphi) = -\frac{1}{a} \sum_{n=1}^{\infty} -\eta_n \beta_n P_n(\cos\varphi)$$

Hence,

$$h_n = \frac{\eta_n \beta_n}{a} \; ; \quad \text{or} \quad \eta_n = \frac{a h_n}{\beta_n} \quad \text{for} \quad n \geq 1$$

It will be sufficient in the sequel to take $\eta_o = 0$. Thus,

$$\frac{d}{d\varphi} T_{\varphi z u}(\varphi) \equiv \tau_{\varphi z u}(\varphi), \quad \text{where} \quad T_{\varphi z u}(\varphi) = \frac{a}{(1+\frac{h}{2a})^3} \sum_{n=1}^{\infty} \frac{h_n}{\beta_n} P_n(\cos\varphi) \tag{6.39}$$

$$\frac{d}{d\varphi} T_{\varphi z \ell}(\varphi) \equiv \tau_{\varphi z \ell}(\varphi), \quad \text{where} \quad T_{\varphi z \ell}(\varphi) = \frac{a}{(1-\frac{h}{2a})^3} \sum_{n=1}^{\infty} \frac{h_n}{\beta_n} P_n(\cos\varphi) \tag{6.40}$$

both to within a constant term.

7. Prebuckling State

It is well known that a complete monocoque spherical shell undergoes only a uniform inward radial deformation prior to buckling.

The corresponding stress distribution is a uniform compressive stress. The complete spherical sandwich shell differs from its monocoque counterpart by its anisotropy. It is desired to determine whether a uniform prebuckling state is also possible for the sandwich sphere, or whether the anisotropy initiates bending before buckling occurs. To this end, it is assumed that $\tau_{\varphi z}$ is zero in the core. Then one determines whether or not the corresponding stress state can be maintained by constant values of w_u and w_ℓ, and zero values of u_u and u_ℓ at the interfaces.

The core equations (6.1) and (6.2) reduce to

$$\frac{\partial}{\partial z}\left[(1+\frac{z}{a})^2 \sigma_z^o \sin\varphi\right] = 0 \tag{7.1}$$

Thus,

$$(1+\frac{z}{a})^2 \sigma_z^o(\varphi,z) \sin\varphi = f^o(\varphi)$$

for some function f^o of φ alone. Then

$$\sigma_z^o(\varphi,z) = E_z \frac{\partial w^o(\varphi,z)}{\partial z} = \frac{f^o(\varphi)}{(1+\frac{z}{a})^2 \sin\varphi}$$

$$w^o(\varphi,z) = -\frac{af^o(\varphi)}{E_z(1+\frac{z}{a})\sin\varphi} + g^o(\varphi) \tag{7.2}$$

for some function g^o of φ alone. The boundary conditions for the uniform prebuckling case are

$$w_u^o = w^o(\varphi, \frac{h}{2}) = -\frac{af^o(\varphi)}{E_z(1 + \frac{h}{2a})\sin\varphi} + g^o(\varphi) \qquad (7.3)$$

$$w_\ell^o = w^o(\varphi, -\frac{h}{2}) = -\frac{af^o(\varphi)}{E_z(1 - \frac{h}{2a})\sin\varphi} + g^o(\varphi) \qquad (7.4)$$

where w_u^o and w_ℓ^o are the constant radial displacements of the upper and lower facings, respectively. Subtracting equation (7.4) from (7.3), one obtains

$$w_u^o - w_\ell^o = -\frac{af^o(\varphi)}{E_z(1 + \frac{h}{2a})\sin\varphi} + \frac{af^o(\varphi)}{E_z(1 - \frac{h}{2a})\sin\varphi}$$

or

$$w_u^o - w_\ell^o = \frac{h}{E_z[1 - (\frac{h}{2a})^2]\sin\varphi} f^o(\varphi) \qquad (7.5)$$

Also,

$$(1 + \frac{h}{2a})w_u^o - (1 - \frac{h}{2a})w_\ell^o = (1 + \frac{h}{2a})g^o(\varphi) - (1 - \frac{h}{2a})g^o(\varphi)$$

or

$$(1 + \frac{h}{2a})w_u^o - (1 - \frac{h}{2a})w_\ell^o = \frac{h}{a}\ g^o(\varphi) \qquad (7.6)$$

By equation (7.5),

$$\frac{f^o(\varphi)}{\sin\varphi} = \frac{E_z[1 - (\frac{h}{2a})^2]}{h}(w_u^o - w_\ell^o)$$

Hence

$$\sigma_z^{\,o}(\varphi, z) = \frac{f^o(\varphi)}{(1+\frac{z}{a})^2 \sin\varphi} = \frac{E_z[1+(\frac{h}{2a})^2]}{h(1+\frac{z}{a})^2}\,(w_u^{\,o} - w_\ell^{\,o})$$

$$\sigma_{zu}^{\,o}(\varphi) \equiv \sigma_z^{\,o}(\varphi, \frac{h}{2}) = \frac{E_z}{h}\;\frac{[1-(\frac{h}{2a})^2]}{[1+\frac{h}{2a}]^2}\,(w_u^{\,o} - w_\ell^{\,o}) \tag{7.7}$$

$$\sigma_{z\ell}^{\,o}(\varphi) \equiv \sigma_z^{\,o}(\varphi, -\frac{h}{2}) = \frac{E_z}{h}\,\frac{[1-(\frac{h}{2a})^2]}{[1-\frac{h}{2a}]^2}\,(w_u^{\,o} - w_\ell^{\,o}) \tag{7.8}$$

The equilibrium equations (4.1), (4.2) and (4.3) for the upper facing in the prebuckling state become

$$-2N_u^{\,o} + a_u P_{zu}^{\,o} = 0, \text{ where}$$

$$N_{\varphi u}^{o} = N_{\theta u}^{o} = N_u^{o} \quad \text{and} \quad P_{zu}^{o} = -\left[\frac{(1+\frac{h+2t_u}{2a})^2}{A_u^{\,2}}\,p + \frac{(1+\frac{h}{2a})^2}{A_u^{\,2}}\,\sigma_{zu}^{o}\right]$$

Define $p^* = (1 + \frac{h + 2t_u}{2a})^2\, p$. Then

$$N_u^{o} = \frac{a_u}{2}\,P_{zu}^{o} = \frac{a_u}{2}\left[\frac{p^* + (1+\frac{h}{2a})^2\,\sigma_{zu}^{\,o}}{A_u^{\,2}}\right] \tag{7.9}$$

By the stress-displacement relation (4.4),

$$N_u^{\,o} = \frac{E_u t_u}{1-\nu_u^{\,2}}\left[(1+\nu_u)\,\frac{w_u^{\,o}}{a_u}\right] = \frac{E_u t_u}{a_u(1-\nu_u)}\,w_u^{\,o} \tag{7.10}$$

Hence, combining equations (4.9) and (4.10),

$$w_u^o = \frac{a_u^{\,2}(1-\nu_u)}{2E_u t_u}\, p_{zu}^{\,o} = -\frac{a^2(1-\nu_u)}{2E_u t_u}\left[p^* + \left(1+\frac{h}{2a}\right)^2 \sigma_{zu}^{\,o}\right] \qquad (7.11)$$

The equilibrium equations (4.1), (4.2), and (4.3) for the lower facing in the prebuckling state become

$$2N_\ell^o + a_\ell p_{z\ell}^{\,o} = 0$$

where

$$N_{\varphi\ell}^{\,o} = N_{\theta\ell}^{\,o} = N_u^o \quad \text{and} \quad p_{z\ell}^{\,o} = \frac{\left(1-\frac{h}{2a}\right)^2}{A_\ell^{\,2}}\, \sigma_{z\ell}^{\,o}$$

Thus, $N_\ell^o = \frac{a_\ell}{2}\, p_{z\ell}^{\,o}$; and again using the Hooke's law relation (4.4),

$$N_\ell^o = \frac{E_\ell t_\ell}{1-\nu_\ell^{\,2}}\left[(1+\nu_\ell)\,\frac{w_\ell^o}{a_\ell}\right] = \frac{E_\ell t_\ell}{a_\ell(1-\nu_\ell)}\, w_\ell^o$$

Combining the two equations for N_ℓ^o,

$$w_\ell^o = \frac{a_\ell^{\,2}(1-\nu_\ell)}{2E_\ell t_\ell}\, p_{z\ell}^{\,o} = \frac{a^2(1-\nu_\ell)}{2E_\ell t_\ell}\left(1-\frac{h}{2a}\right)^2 \sigma_{z\ell}^{\,o} \qquad (7.12)$$

Applying equations (7.7) and (7.8) for $\sigma_{zu}^{\,o}$ and $\sigma_{z\ell}^{\,o}$ in terms of $(w_u^o - w_\ell^o)$, one obtains from equations (7.11) and (7.12) the following system of equations for w_u^o and w_ℓ^o:

$$w_u^o = -\frac{a^2(1-\nu_u)}{2E_u t_u}\left\{p^* + \frac{E_z}{h}\left[1-\left(\frac{h}{2a}\right)^2\right](w_u^o - w_\ell^o)\right\} \qquad (7.13)$$

$$w_\ell^o = \frac{a^2(1-\nu_\ell)}{2E_\ell t_\ell}\left\{\frac{E_z}{h}\left[1-\left(\frac{h}{2a}\right)^2\right](w_u^o - w_\ell^o)\right\} \qquad (7.14)$$

It will suffice to solve this system for $(w_u^o - w_\ell^o)$, since only the prebuckling normal stresses σ_{zu}^o and σ_{zl}^o will be required in the facing analyses. Subtracting equation (7.14) from equation (7.13), one obtains

$$(w_u^o - w_\ell^o) = -\frac{a^2(1-\nu_u)}{2E_u t_u}\,p^* - \frac{E_z}{h}(w_u^o - w_\ell^o)\left[1-\left(\frac{h}{2a}\right)^2\right]\left[\frac{a^2(1-\nu_u)}{2E_u t_u} + \frac{a^2(1-\nu_\ell)}{2E_\ell t_\ell}\right]$$

Hence,

$$w_u^o - w_\ell^o = -\frac{a^2(1-\nu_u)}{2E_u t_u}\,p^* \cdot \frac{1}{\left\{1+\frac{E_z}{h}\left[1-\left(\frac{h}{2a}\right)^2\right]\left[\frac{a^2(1-\nu_u)}{2E_u t_u}+\frac{a^2(1-\nu_\ell)}{2E_\ell t_\ell}\right]\right\}}$$

Let

$$q^o = \frac{a^2(1-\nu_u)}{2E_u t_u}\,\frac{}{\left\{1+\frac{E_z}{h}\left[1-\left(\frac{h}{2a}\right)^2\right]\left[\frac{a^2(1-\nu_u)}{2E_u t_u}+\frac{a^2(1-\nu_\ell)}{2E_\ell t_\ell}\right]\right\}}$$

Then, from equations (7.7) and (7.8),

$$\sigma_{zu}^o(\varphi) = -\frac{E_z}{h}\,\frac{\left[1-\left(\frac{h}{2a}\right)^2\right]}{\left[1+\frac{h}{2a}\right]^2}\,q^o\,p^* \qquad (7.16)$$

$$\sigma_{z\ell}(\varphi) = -\frac{E_z}{h}\frac{[1-(\frac{h}{2a})^2]}{[1-\frac{h}{2a}]^2}\; q^o p^* \tag{7.17}$$

Similarly, the prebuckling facing compressive stresses are

$$N_u^o = -\frac{a_u}{2A_u^2}\left\{p^* - \frac{E_z}{h}\frac{[1-(\frac{h}{2a})^2]}{[1+\frac{h}{2a}]^2}\, q^o p^*\right\} = -\frac{a_u p^*}{2A_u^2}\left\{1 - \frac{E_z}{h}\frac{[1-(\frac{h}{2a})^2]}{[1+\frac{h}{2a}]^2}\, q^o\right\} \tag{7.18}$$

$$N_\ell^o = \frac{a_\ell}{2A_\ell^2}\left\{-\frac{E_z}{h}\frac{[1-(\frac{h}{2a})^2]}{[1-\frac{h}{2a}]^2}\, q^o p^*\right\} = -\frac{a_\ell p^*}{2A_\ell^2}\left\{\frac{E_z}{h}\frac{[1-(\frac{h}{2a})^2]}{[1-\frac{h}{2a}]^2}\, q^o\right\} \tag{7.19}$$

The existence of the uniform prebuckling state has thus been proved. Although the individual facings are momentless in the prebuckling state, the composite shell does experience a bending moment. This is due to the difference between N_u^o and N_ℓ^o. It is easily seen that in the limit as $E_z \to \infty$ and $h \to 0$, N_u^o and N_ℓ^o approach the common value $-\frac{pa}{4}$. Thus, the proper reduction to the monocoque case is obtained.

8. Incipient Buckling

The critical state of equilibrium where buckling is incipient is now analyzed. Incremental buckling deformations are superposed upon the uniform prebuckling state.

The lowest external pressure at which nonzero buckling deformations are possible is the critical pressure.

Upper facing analysis

$$N_{\varphi u} = N_u^o + N'_{\varphi u}, \quad N_{\theta u} = N_u^o + N'_{\theta u}; \quad P_{zu} = P_{zu}^o - \frac{p^*}{A_u^2}(\varepsilon_1 + \varepsilon_2) - \frac{(1 + \frac{h}{2a})^2}{A_u^2}\sigma_{zu};$$

$$P_{\varphi u} = - \frac{(1 + \frac{h}{2a})^2 \tau_{\varphi zu}}{A_u^2} \quad \text{and} \quad R_{\theta u} = \frac{t_u(1 + \frac{h}{2a})^2}{2A_u^2}\tau_{\varphi zu}$$

Thus, defining

$$\tau^*_{\varphi zu} = \frac{(1 + \frac{h}{2a})^2}{A_u^2}\tau_{\varphi zu}$$

$$\sigma^*_{zu} = \frac{(1 + \frac{h}{2a})^2}{A_u^2}\sigma_{zu}$$

the equilibrium equations (4.1), (4.2) and (4.3) become

$$\frac{d}{d\varphi}[N_u^o + N'_{\varphi u}] + [N_u^o + N'_{\varphi u} - N_u^o - N'_{\theta u}] \cot\varphi + Q_{\varphi u}$$
$$+ (N_u^o + N'_{\theta u})(\frac{u_u}{a_u} - \frac{dw_u}{a_u d\varphi})$$
$$+ Q_{\varphi u}(\frac{du_u}{a_u d\varphi} - \frac{d^2 w_u}{a_u d\varphi^2}) - a_u \tau^*_{\varphi zu} = 0 \tag{8.1}$$

$$\frac{dQ_{\varphi u}}{d\varphi} + Q_{\varphi u} \cot \varphi - [N_u^o + N'_{\varphi u} + N_u^o + N'_{\theta u}] - (N_u^o + N'_{\varphi u}) \frac{d}{d\varphi} (\frac{u_u}{a_u}$$

$$- \frac{dw_u}{a_u d\varphi}) - (N_u^o + N'_{\theta u})(\frac{u_u}{a_u} - \frac{dw_u}{a_u d\varphi}) \cot \varphi + [\frac{2N_u^o}{a_u} - \frac{p^*}{A_u^2} (\frac{du_u}{a_u d\varphi}$$

$$+ \frac{u_u \cot \varphi}{a_u} + \frac{2w_u}{a_u}) - \sigma^*_{zu}] = 0 \tag{8.2}$$

$$\frac{dM_{\varphi u}}{d\varphi} + (M_{\varphi u} - M_{\theta u}) \cot \varphi + M_{\theta u}(\frac{u_u}{a_u} - \frac{dw_u}{a_u d\varphi}) - a_u Q_{\varphi u} + \frac{a_u t_u}{2} \tau^*_{\varphi zu} = 0 \tag{8.3}$$

The equations (8.1) - (8.3) are nonlinear; they are now linearized as in the customary monocoque sphere analysis. Then solving for $Q_{\varphi u}$ in equation (8.3),

$$Q_{\varphi u} = \frac{1}{a_u} \frac{dM_{\varphi u}}{d\varphi} + \frac{(M_{\varphi u} - M_{\theta u})}{a_u} \cot \varphi + \frac{t_u}{2} \tau^*_{\varphi zu} \tag{8.4}$$

Substituting equation (8.4) in equations (8.1) and (8.2), one obtains the <u>linearized</u> governing equations for the top facing:

$$\frac{dN'_{\varphi u}}{d\varphi} + [N'_{\varphi u} - N'_{\theta u}] \cot \varphi + \frac{dM_{\varphi u}}{a_u d\varphi} + \frac{(M_{\varphi u} - M_{\theta u})}{a_u} \cot \varphi$$

$$+ N_u^o (\frac{u_u}{a_u} - \frac{dw_u}{a_u d\varphi}) - \tau^*_{\varphi zu} (a_u - \frac{t_u}{2}) = 0 \tag{8.5}$$

$$[\frac{d}{d\varphi} + \cot \varphi] \left\{ \frac{dM_{\varphi u}}{a_u d\varphi} + \frac{(M_{\varphi u} - M_{\theta u})}{a_u} \cot \varphi + \frac{t_u}{2} \tau^*_{\varphi zu} \right\}$$

$$- (N'_{\varphi u} + N'_{\theta u}) - N_u^o [\frac{d}{d\varphi} + \cot \varphi] \left\{ \frac{u_u}{a_u} - \frac{dw_u}{a_u d\varphi} \right\} - a_u \sigma^*_{zu} -$$

$$\frac{a_u p^*}{A_u^2}\left[\frac{du_u}{a_u d\varphi} + \frac{u_u \cot\varphi}{a_u} + \frac{2w_u}{a_u}\right] = 0 \tag{8.6}$$

Lower facing analysis

$$N_{\varphi\ell} = N_\ell^o + N'_{\varphi\ell}\ ; \quad N_{\theta\ell} = N_\ell^o + N'_{\theta\ell}; \quad P_{z\ell} = P^o_{z\ell} + \frac{(1 - \frac{h}{2a})^2}{A_\ell^2}\sigma_{z\ell}$$

$$P_{\varphi\ell} = \frac{(1 - \frac{h}{2a})^2}{A_\ell^2}\tau_{\varphi z\ell} \quad \text{and} \quad R_{\theta\ell} = \frac{t_\ell}{2}\,\frac{(1 - \frac{h}{2a})^2}{A_\ell^2}\tau_{\varphi z\ell}$$

Thus, defining

$$\tau^*_{\varphi z\ell} = \frac{(1 - \frac{h}{2a})^2}{A_\ell^2}\tau_{\varphi z\ell}$$

$$\sigma^*_{z\ell} = \frac{(1 - \frac{h}{2a})^2}{A_\ell^2}\sigma_{z\ell}$$

the equilibrium equations (4.1), (4.2) and (4.3) become

$$\frac{d}{d\varphi}\left[N_\ell^o + N'_{\varphi\ell}\right] + \left[N_\ell^o + N'_{\varphi\ell} - N_\ell^o - N_{\theta\ell}\right]\cot\varphi + Q_{\varphi\ell}$$

$$+ (N_\ell^o + N'_{\theta\ell})\left(\frac{u_\ell}{a_\ell} - \frac{dw_\ell}{a_\ell d\varphi}\right) + Q_{\varphi\ell}\left(\frac{du_\ell}{a_\ell d\varphi} - \frac{d^2 w_\ell}{a_\ell d\varphi^2}\right)$$

$$+ a_\ell\,\tau^*_{\varphi z\ell} = 0 \tag{8.7}$$

$$\frac{dQ_{\varphi\ell}}{d\varphi} + Q_{\varphi\ell}\cot\varphi - \left[N_\ell^o + N'_{\varphi\ell} + N_\ell^o + N'_{\theta\ell}\right] - (N_\ell^o +$$

$$+ N'_{\varphi\ell}) \frac{d}{d\varphi} \left(\frac{u_\ell}{a_\ell} - \frac{dw_\ell}{a_\ell d\varphi}\right) - (N_\ell^o + N'_{\theta\ell})\left(\frac{u_\ell}{a_\ell} - \frac{dw_\ell}{a_\ell d\varphi}\right) \cot\varphi$$

$$+ a_\ell \left[\frac{2N_\ell^o}{a_\ell} + \sigma^*_{z\ell} \right] = 0 \tag{8.8}$$

$$\frac{dM_{\varphi\ell}}{d\varphi} + (M_{\varphi\ell} - M_{\theta\ell}) \cot\varphi + M_{\theta\ell} \left(\frac{u_\ell}{a_\ell} - \frac{dw_\ell}{a_\ell d\varphi}\right)$$

$$- a_\ell Q_{\varphi\ell} + \frac{a_\ell t_\ell}{2} \tau^*_{\varphi z\ell} = 0 \tag{8.9}$$

Linearizing these equations, solving for $Q_{\varphi\ell}$ in equation (8.9) and substituting in the remaining equations (8.7) and (8.8), one gets the following governing equations for the lower facing:

$$\frac{dN'_{\varphi\ell}}{d\varphi} + [N'_{\varphi\ell} - N'_{\theta\ell}] \cot\varphi + \frac{dM_{\varphi\ell}}{a_\ell d\varphi} + \frac{(M_{\varphi\ell} - M_{\theta\ell})}{a_\ell} \cot\varphi$$

$$+ N_\ell^o \left[\frac{u_\ell}{a_\ell} - \frac{dw_\ell}{a_\ell d\varphi} \right] + \tau^*_{\varphi z\ell} \left(a_\ell + \frac{t_\ell}{2}\right) = 0 \tag{8.10}$$

$$\left(\frac{d}{d\varphi} + \cot\varphi\right) \left\{ \frac{dM_{\varphi\ell}}{a_\ell d\varphi} + \frac{(M_{\varphi\ell} - M_{\theta\ell})}{a_\ell} \cot\varphi + \frac{t_\ell}{2} \tau^*_{\varphi z\ell} \right.$$

$$\left. - (N'_{\varphi\ell} + N'_{\theta\ell}) - N_\ell^o \left[\frac{d}{d\varphi} + \cot\varphi\right] \left\langle \frac{u_\ell}{a_\ell} - \frac{dw_\ell}{a_\ell d\varphi} \right\rangle \right\} + a_\ell \sigma^*_{z\ell} = 0 \tag{8.11}$$

Hence, to solve the spherical shell buckling problem, it is necessary to solve the four differential equations (8.5), (8.6), (8.10) and (8.11) subject to the Hooke's law relations (4.4) - (4.7) for each facing. Note that equations (8.5) and (8.10) differ only in the loading terms involving shear stresses at the interfaces; likewise, equations (8.5) and (8.11) differ only in the loading terms. Hence, one may treat the main terms in the two equations alike, devoting special attention only to the loading terms.

Substituting the Hooke's law relations (4.4) - (4.7) into the common terms

$$\frac{dN'_\varphi}{d\varphi} + [N'_\varphi - N'_\theta] \cot\varphi + \frac{dM_\varphi}{a d\varphi} + \frac{(M_\varphi - M_\theta)}{a} \cot\varphi + N^o\left(\frac{u}{a} - \frac{dw}{a d\varphi}\right) \quad (8.12)$$

one obtains

$$\frac{Et}{a(1-\nu^2)} \left\{ (1+\alpha)\left[\frac{d^3\psi}{d\varphi^3} + \frac{d^2\psi}{d\varphi^2}\cot\varphi - (\nu + \cot^2\varphi)\frac{d\psi}{d\varphi}\right] \right. \quad (8.13)$$

$$+ (1+\nu)\frac{dw}{d\varphi} - \alpha\left[\frac{d^3w}{d\varphi^3} + \frac{d^2w}{d\varphi^2}\cot\varphi\right.$$

$$\left.\left. - (\nu + \cot^2\varphi)\frac{dw}{d\varphi}\right] + \phi\left[\frac{d\psi}{d\varphi} - \frac{dw}{d\varphi}\right]\right\}$$

where $\alpha = \frac{D(1-\nu^2)}{a^2 Et} = \frac{t^2}{12a^2}$, $\phi = \frac{N^o(1-\nu^2)}{Et}$ and, as before, ψ is a potential function such that $u = \frac{d\psi}{d\varphi}$.

Introducing the differential operator H defined by

$$H(\cdot) = \frac{d^2(\cdot)}{d\varphi^2} + \frac{d(\cdot)}{d\varphi}\cot\varphi + 2(\cdot) \quad (8.14)$$

expression (8.13) becomes

$$\left\{(1+\alpha)\frac{d}{d\varphi}[H(\psi)-(1+\nu)\psi]+\frac{d}{d\varphi}\left\{(1+\nu)w-\alpha H(w)\right.\right.$$

$$\left.\left.+\alpha(1+\nu)w+\varphi(\psi-w)\right\}\right\}\frac{Et}{a(1-\nu^2)} \qquad (8.15)$$

Hence, defining $\alpha_u = \dfrac{D_u(1-\nu_u^2)}{a_u^2\, E_u t_u} = \dfrac{t_u^2}{12a_u^2}$; $\varphi_u = \dfrac{N_u^o(1-\nu_u^2)}{E_u t_u}$

$$\alpha_\ell = \frac{D_\ell(1-\nu_\ell^2)}{a_\ell^2 E_\ell t_\ell} = \frac{t_\ell^2}{12a_\ell^2} \ ; \quad \text{and} \quad \varphi_\ell = \frac{N_\ell^o(1-\nu_\ell^2)}{E_\ell t_\ell} \ ,$$

equations (8.5) and (8.10) become, respectively,

$$(1+\alpha_u)\frac{d}{d\varphi}[H(\psi_u)-(1+\nu_u)\psi_u]+\frac{d}{d\varphi}\left\{(1+\nu_u)w_u-\alpha_u H(w_u)\right.$$

$$\left.+\alpha_u(1+\nu_u)w_u+\varphi_u(\psi_u-w_u)\right\}-(a_u-\frac{t_u}{2})\frac{a_u(1-\nu_u^2)}{E_u t_u}\tau^*_{\varphi zu}=0$$

$$(8.16)$$

$$(1+\alpha_\ell)\frac{d}{d\varphi}[H(\psi_\ell)-(1+\nu_\ell)\psi_\ell]+\frac{d}{d\varphi}\left\{(1+\nu_\ell)w_\ell-\alpha_\ell H(w_\ell)\right.$$

$$\left.+\alpha_\ell(1+\nu_\ell)w_\ell+\varphi_\ell(\psi_\ell-w_\ell)\right\}+(a_\ell+\frac{t_\ell}{2})\frac{a_\ell(1-\nu_\ell^2)}{E_\ell t_\ell}\tau^*_{\varphi z\ell}=0.$$

$$(8.17)$$

But, by equations (6.5), (6.39) and (6.40),

$$(a_u-\frac{t_u}{2})\tau^*_{\varphi zu}=\frac{a(1+\frac{h}{2a})^3}{A_u^2}\tau_{\varphi zu}=\frac{a}{A_u^2}\tau_{\varphi zm}$$

$$(a_\ell+\frac{t_\ell}{2})\tau^*_{\varphi z\ell}=\frac{a(1-\frac{h}{2a})^3}{A_\ell^2}\tau_{\varphi z\ell}=\frac{a}{A_\ell^2}\tau_{\varphi zm}$$

where $\tau_{\varphi zm} = \tau(\varphi,z)\Big]_{z=o}$

Since $\dfrac{dT_{\varphi z}}{d\varphi} = \tau_{\varphi z}$, then

$$\tau_{\varphi zm} = \frac{dT_{\varphi zm}}{d\varphi}$$

where

$$T_{\varphi zm} = T_{\varphi z}(\varphi,0) = a \sum_{n=1}^{\infty} \frac{h_n}{\beta_n} P_n \ (\cos \varphi)$$

using equations (6.39) and (6.40). Thus,

$$(a_u - \frac{t_u}{2})\tau^*_{\varphi zu} = \frac{a}{A_u^{\ 2}} \frac{d}{d\varphi} T_{\varphi zm}$$

$$(a_\ell + \frac{t_\ell}{2})\tau^*_{\varphi z\ell} = \frac{a}{A_\ell^{\ 2}} \frac{d}{d\varphi} T_{\varphi zm}$$

Hence, equations (8.16) and (8.17) become

$$\frac{d}{d\varphi}\left\{(1+\alpha_u)[H(\psi_u) - (1+\nu_u)\psi_u] + (1+\nu_u)w_u - \alpha_u H(w_u) + \alpha_u(1+\nu_u)w_u + \varphi_u(\psi_u - w_u) - \frac{a}{A_u^{\ 2}}\left[\frac{a_u(1-\nu_u^{\ 2})}{E_u t_u}\right] T_{\varphi zm}\right\} = 0 \qquad (8.18)$$

$$\frac{d}{d\varphi}\left\{(1+\alpha_\ell)[H(\psi_\ell) - (1+\nu_\ell)\psi_\ell] + (1+\nu_\ell)w_\ell - \alpha_\ell H(w_\ell) + \alpha_\ell(1+\nu_\ell)w_\ell + \varphi_\ell(\psi_\ell - w_\ell) + \frac{a}{A_\ell^{\ 2}}\left[\frac{a_\ell(1-\nu_\ell^{\ 2})}{E_\ell t_\ell}\right] T_{\varphi zm}\right\} = 0 \qquad (8.19)$$

The addition of an arbitrary constant to each of the displacement potentials ψ_u and ψ_ℓ will not affect the corresponding displacement functions u_u and u_ℓ. Hence, equations (8.18) and (8.19) may be integrated with respect to φ, and the constants of integration taken as zero. This corresponds to the addition of arbitrary constants to ψ_u and ψ_ℓ. Hence, equations (8.18) and (8.19) become

$$(1 + \alpha_u)[H(\psi_u) - (1+\nu_u)\psi_u] + (1+\nu_u)w_u - \alpha_u H(w_u) + \alpha_u(1+\nu_u)w_u$$

$$+ \varphi_u(\psi_u - w_u) - \frac{a}{A_u^2}\left[\frac{a_u(1-\nu_u^2)}{E_u t_u}\right]T_{\varphi zm} = 0 \tag{8.20}$$

$$(1 + \alpha_\ell)[H(\psi_\ell) - (1+\nu_\ell)\psi_\ell] + (1+\nu_\ell)w_\ell - \alpha_\ell H(w_\ell) + \alpha_\ell(1+\nu_\ell)w_\ell$$

$$+ \varphi_\ell(\psi_\ell - w_\ell) + \frac{a}{A_\ell^2}\left[\frac{a_\ell(1-\nu_\ell^2)}{E_\ell t_\ell}\right]T_{\varphi zm} = 0 \tag{8.21}$$

Similarly, substitution of the Hooke's law relations (4.4) - (4.7) into the expression

$$\left(\frac{d}{d\varphi} + \cot\varphi\right)\left\{\frac{dM_\varphi}{a d\varphi} + \frac{\cot\varphi}{a}(M_\varphi - M_\theta)\right\}$$

$$- (N_\varphi' + N_\theta') - N^0\left[\frac{d}{d\varphi} + \cot\varphi\right]\left\{\frac{u}{a} - \frac{dw}{a d\varphi}\right\} \tag{8.22}$$

yields the following:

$$\frac{Et}{a(1-\nu^2)}\left\{\alpha\left[\left(\frac{d^4\psi}{d\varphi^4} + 2\frac{d^3\psi}{d\varphi^3}\cot\varphi - (1+\nu+\cot^2\varphi)\frac{d^2\psi}{d\varphi^2} + \right.\right.\right.$$

$$+ (2 - \nu + \cot^2 \varphi) \frac{d\psi}{d\varphi} \cot \varphi - \frac{d^4 w}{d\varphi^4} + 2 \frac{d^3 w}{d\varphi^3} \cot \varphi - (1 + \nu + \cot^2 \varphi) \frac{d^2 w}{d\varphi^2}$$

$$+ (2 - \nu + \cot^2 \varphi) \frac{dw}{d\varphi} \cot \varphi \Bigg] - (1 + \nu) \left[\frac{d^2 \psi}{d\varphi^2} + \cot \varphi \frac{d\psi}{d\varphi} + 2w \right]$$

$$- \varphi \left[\frac{d^2 \psi}{d\varphi^2} + \frac{d\psi}{d\varphi} \cot \varphi - \left(\frac{d^2 w}{d\varphi^2} + \frac{dw}{d\varphi} \cot \varphi \right) \right] \Bigg\} \tag{8.23}$$

It follows from the definition (8.14) of the operator H that

$$HH(\cdot) = \frac{d^4(\cdot)}{d\varphi^4} + 2 \frac{d^3(\cdot)}{d\varphi^3} \cot \varphi + (2 - \cot^2 \varphi) \frac{d^2}{d\varphi^2}$$

$$+ [5 + \cot^2 \varphi] \frac{d(\cdot)}{d\varphi} \cot \varphi + 4(\cdot)$$

Thus, expression (8.23) becomes

$$\frac{Et}{a(1 - \nu^2)} \Big\{ \alpha [HH(\psi - w) - (3 + \nu) H(\psi - w) + 2(1 + \nu)(\psi - w)]$$

$$- (1 + \nu) H(\psi) + 2(1 + \nu)(\psi - w) - \varphi H(\psi - w) + 2\varphi(\psi - w) \Big\} \tag{8.24}$$

Thus, equation (8.6) for the upper facing becomes

$$\alpha_u HH(\psi_u - w_u) - (3 + \nu_u)\alpha_u H(\psi_u - w_u) + 2(1 + \nu_u) \alpha_u (\psi_u - w_u)$$

$$- (1 + \nu_u) H(\psi_u) + 2(1 + \nu_u)(\psi_u - w_u) - \varphi_u H(\psi_u - w_u) + 2\varphi_u(\psi_u - w_u)$$

$$+ \frac{a_u(1 - \nu_u^2)}{E_u t_u} \cdot \frac{t_u}{2} \left[H(T^*_{\varphi zu}) - 2T^*_{\varphi zu} \right] - \frac{a_u(1 - \nu_u^2)}{E_u t_u A_u^2} p^* \Big[H(\psi_u)$$

$$- 2(\psi_u - w_u) \Big] - \frac{a_u^2 (1 - \nu_u^2)}{E_u t_u} \sigma^*_{zu} = 0, \quad \text{where } T^*_{\varphi zu} = \frac{(1 + \frac{h}{2a})^2}{A_u^2} T_{\varphi zu}$$

(8.30)

Similarly, equation (8.11) for the top facing becomes

$$\alpha_\ell HH(\psi_\ell - w_\ell) - (3+\nu_\ell)\alpha_\ell H(\psi_\ell - w_\ell) + 2(1+\nu_\ell)\alpha_\ell(\psi_\ell - w_\ell)$$

$$- (1+\nu_\ell)H(\psi_\ell) + 2(1+\nu_\ell)(\psi_\ell - w_\ell) - \varphi_\ell H(\psi_\ell - w_\ell) + 2\varphi_\ell(\psi_\ell - w_\ell)$$

$$+ \frac{a_\ell(1-\nu_\ell^2)}{E_\ell t_\ell} \circ \frac{t_\ell}{2}\left[H(T^*_{\varphi z\ell}) - 2T^*_{\varphi z\ell}\right] + \frac{a_\ell^2(1-\nu_\ell^2)}{E_\ell t_\ell}\sigma^*_{z\ell} = 0 \quad (8.3)$$

where $T^*_{\varphi z\ell} = \dfrac{(1 - \frac{h}{2a})^2}{A_\ell^2} T_{\varphi z\ell}$.

To determine the buckling pressure for the spherical sandwich shell, equations (8.20), (8.21), (8.30) and (8.31) must be solved simultaneously, subject to the expansions (6.32), for the unknown displacement functions.

9. Solution of the Buckling Problem

The system of equations (8.20), (8.30), (8.21) and (8.31), respectively, becomes

$$(1+\alpha_u)\left[H(\psi_u) - (1+\nu_u)\psi_u\right] + (1+\nu_u)(1+\alpha_u)w_u$$

$$- \alpha_u H(w_u) + \varphi_u(\psi_u - w_u) - \frac{a}{A_u^2}\left[\frac{a_u(1-\nu_u^2)}{E_u t_u}\right]T_{\varphi zm} = 0 \qquad (9.1)$$

$$\alpha_u HH(\psi_u - w_u) - (3+\nu_u)\alpha_u H(\psi_u - w_u) + 2(1+\nu_u)(1+\alpha_u)(\psi_u - w_u)$$

$$- (1+\nu_u)H(\psi_u) - \varphi_u H(\psi_u - w_u) + 2\varphi_u(\psi_u - w_u) + \frac{a_u(1-\nu_u^2)}{2E_u}\left[H(T^*_{\varphi zu})\right.$$

$$\left.- 2T^*_{\varphi zu}\right] - \frac{a_u(1-\nu_u^2)}{E_u t_u}\frac{p^*}{A_u^2}\left[H(\psi_u) - 2(\psi_u - w_u)\right] - \frac{a_u^2(1-\nu_u^2)}{E_u t_u}\sigma^*_{zu} = 0 \quad (9.2)$$

$$(1 + \alpha_\ell)[H(\psi_\ell) - (1 + \nu_\ell)\psi_\ell] + (1 + \nu_\ell)(1 + \alpha_\ell)w_\ell - \alpha_\ell H(w_\ell)$$

$$+ \varphi_\ell(\psi_\ell - w_\ell) + \frac{a}{A_\ell^2}\left[\frac{a_\ell(1 - \nu_\ell^2)}{E_\ell t_\ell}\right] T_{\varphi zm} = 0 \qquad (9.3)$$

$$\alpha_\ell HH(\psi_\ell - w_\ell) - (3 + \nu_\ell)\alpha_\ell H(\psi_\ell - w_\ell) + 2(1+\nu_\ell)(1+\alpha_\ell)(\psi_\ell - w_\ell)$$

$$- (1 + \nu_\ell) H(\psi_\ell) - \varphi_\ell H(\psi_\ell - w_\ell) + 2\varphi_\ell(\psi_\ell - w_\ell)$$

$$+ \frac{a_\ell(1 - \nu_\ell^2)}{2E_\ell}[H(T^*_{\varphi z\ell}) - 2T^*_{\varphi z\ell}] + \frac{a_\ell^2(1 - \nu_\ell^2)}{E_\ell t_\ell}\sigma^*_{z\ell} = 0 \qquad (9.4)$$

Now, define

$$\beta_u = \frac{a^2(1 - \nu_u^2)}{E_u t_u}$$

$$\beta_\ell = \frac{a^2(1 - \nu_\ell^2)}{E_\ell t_\ell}$$

$$\delta_u = \frac{a^2(1 - \nu_u^2)}{2 A_u E_u(1 + \frac{h}{2a})} \qquad (9.5)$$

$$\delta_\ell = \frac{a^2(1 - \nu_\ell^2)}{2A_\ell E_\ell(1 - \frac{h}{2a})}$$

Also, define

$$k_1 = \frac{t_u}{4a_u} + \frac{a}{h} - \frac{1}{2} - \frac{t_u}{2hA_u}$$

$$k_2 = -\left(\frac{t_\ell}{4a_\ell} + \frac{a}{h} + \frac{1}{2} + \frac{t_\ell}{2hA_\ell}\right)$$

$$k_3 = \frac{1}{2} - \frac{t_u}{4a_u} + \frac{t_u}{2hA_u} \qquad (9.6)$$

$$k_4 = \frac{1}{2} + \frac{t_\ell}{4a_\ell} + \frac{t_\ell}{2hA_\ell}$$

$$\gamma_n = \frac{3[1 - (\frac{h}{2a})^2]^2}{a^2\left\{\frac{2}{G_z} - \frac{\beta_n}{E_z} + (\frac{1}{G_z} + \frac{\beta_n}{G_z})[1 + (\frac{h}{2a})^2]\right\}}$$

Then, from solution (6.35) for the Fourier-Legendre coefficient h_n of the function h,

$$h_n = \gamma_n\beta_n(k_1a_n + k_2b_n + k_3c_n + k_4d_n) \qquad (9.7)$$

Also, defining

$$\varepsilon_n = \frac{h}{2[1 - (\frac{h}{2a})^2]}\gamma_n\beta_n$$

$$\xi_1 = [1 - (\frac{h}{2a})^2]\frac{E_z}{h}$$

the Fourier-Legendre expansions for σ_{zu} and $\sigma_{z\ell}$ given in equations (6.37) and (6.38) become

$$\sigma_{zu}(\varphi) = \frac{1}{(1 + \frac{h}{2a})^2}\sum_{n=o}^{\infty}\left\{\left[\varepsilon_n(k_1a_n + k_2b_n + k_3c_n + k_4d_n) + \xi_1(c_n - d_n)\right]P_n(\cos\varphi)\right\} \qquad (9.9)$$

$$\sigma_{z\ell} = \frac{1}{\left(1 - \frac{h}{2a}\right)^2} \sum_{n=0}^{\infty} \left\{ \left[-\varepsilon_n (k_1 a_n - k_2 b_n + k_3 c_n + k_4 d_n) + \xi_1 (c_n - d_n) \right] P_n(\cos\psi) \right\} \qquad (9.10)$$

Substitute the Fourier-Legendre expansions of the displacement functions ψ_u, ψ_ℓ, w_u and w_ℓ, and the solutions for the interface core stresses in terms of these functions into the system of equations (9.1) - (9.4). Using the completeness property of the set of Legendre polynomials $\{P_n(\cos\psi)\}$, the following system of homogeneous linear algebraic equations is obtained, where $\lambda_n = n(n+1) - 2$:

$$(1 + \alpha_u)[-\lambda_n a_n - (1 + \nu_u) a_n] + (1 + \nu_u)(1 + \alpha_u) c_n + \alpha_u \lambda_n c_n + \varphi_u (a_n - c_n) - \frac{\beta_u a}{A_u} \gamma_n (k_1 a_n + k_2 b_n + k_3 c_n + k_4 d_n) = 0 \qquad (9.11)$$

$$\begin{aligned} &\alpha_u \lambda_n^2 (a_n - c_n) + (3 + \nu_u)\alpha_u \lambda_n (a_n - c_n) + 2(1 - \nu_u)(1 + \alpha_u)(a_n - c_n) \\ &+ (1 + \nu_u)\lambda_n a_n + \varphi_u \lambda_n (a_n - c_n) + 2\lambda_n (a_n - c_n) \\ &- \delta_u (\lambda_n + 2)\, \Upsilon_n (k_1 a_n + k_2 b_n + k_3 c_n + k_4 d_n) \\ &- \frac{\beta_u p^*}{a_u} [-\lambda_n a_n - 2(a_n - c_n)] - \beta_u [\varepsilon_n (k_1 a_n + k_2 b_n + k_3 c_n \\ &- k_4 d_n) + \xi_1 (c_n - d_n)] = 0 \end{aligned} \qquad (9.12)$$

$$(1 + \alpha_\ell)[-\lambda_n b_n - (1 + \nu_\ell) b_n] + (1 + \nu_\ell)(1 + \alpha_\ell) a_n + \alpha_\ell \lambda_n d_n$$

$$+ \varphi_\ell (b_n - d_n) + \frac{a\ \beta_\ell}{A_\ell} \gamma_n (k_1 a_n + k_2 b_n + k_3 c_n + k_4 d_n) = 0 \qquad (9.13)$$

$$\alpha_\ell \lambda_n^2 (b_n - d_n) + (3 + \nu_\ell) \alpha_\ell \lambda_n (b_n - d_n) + 2(1 + \nu_\ell)(1 + \alpha_\ell)(b_n - d_n)$$

$$+ (1 + \nu_\ell) \lambda_n b_n + \varphi_\ell \lambda_n (b_n - d_n) + 2\varphi_\ell (b_n - d_n)$$

$$- \delta_\ell (\lambda_n + 2)\ \gamma_n (k_1 a_n + k_2 b_n + k_3 c_n + k_4 d_n)$$

$$+ \beta_\ell [-\varepsilon_n (k_1 a_n + k_2 b_n + k_3 c_n + k_4 d_n) + \xi_1 (c_n - d_n)] = 0 \qquad (9.14)$$

In matrix form, the system of equations (9.11) - (9.14) appears as

$$
\begin{bmatrix}
\begin{array}{l} -(1+\alpha_u)(\lambda_n+1+\nu_u)+\varphi_u \\ -\dfrac{\beta_u a}{A_u}\,\gamma_n k_1 \end{array} &
\begin{array}{l} (1+\nu_u)(1+\alpha_u)+\alpha_u\lambda_n-\varphi_u \\ -\dfrac{\beta_u a\gamma_n k_3}{A_u} \end{array} &
-\dfrac{\beta_u a}{A_u}\,\gamma_n k_2 &
-\dfrac{\beta_u a\gamma_n k_4}{A_u} \\[2ex]
\begin{array}{l} \alpha_u\lambda_n^2+(3+\nu_u)\alpha_u\lambda_n \\ +\,2(1+\nu_u)(1+\alpha_u) \\ +\,(1+\nu_u)\lambda_n+\varphi_u\lambda_n+2\varphi_u \\ -\delta_u(\lambda_n+2)\gamma_n k_1+\dfrac{\beta_u p^*}{a_u}\cdot \\ \cdot\,(\lambda_n+2)-\beta_u\varepsilon_n k_1 \end{array} &
\begin{array}{l} -\alpha_u\lambda_n^2-(3+\nu_u)\alpha_u\lambda_n \\ -2(1+\nu_u)(1+\alpha_u) \\ -\varphi_u\lambda_n-2\varphi_u-\delta_u(\lambda_n+2)\gamma_n k_3 \\ -2\,\dfrac{\beta_u p^*}{a_u}-\beta_u\varepsilon_n k_3-\beta_u\xi_1 \end{array} &
\begin{array}{l} -\delta_u(\lambda_n+2)\gamma_n k_2 \\ -\beta_u\varepsilon_n k_2 \end{array} &
\begin{array}{l} -\delta_u(\lambda_n+2)\gamma_n k_4 \\ -\beta_u\varepsilon_n k_4+\beta_u\xi_1 \end{array} \\[2ex]
\dfrac{a\beta_\ell\gamma_n k_1}{A_\ell} &
\dfrac{a\beta_\ell\gamma_n}{A_\ell}\,k_3 &
\begin{array}{l} -(1+\alpha_\ell)(\lambda_n+1+\nu_\ell) \\ +\,\varphi_\ell+\dfrac{a\beta_\ell}{A_\ell}\,\gamma_n k_2 \end{array} &
\begin{array}{l} +(1+\nu_\ell)(1+\alpha_\ell) \\ +\alpha_\ell\lambda_n-\varphi_\ell+\dfrac{a\beta_\ell\gamma_n}{A_\ell}\,k_4 \end{array} \\[2ex]
\begin{array}{l} -\delta_\ell(\lambda_n+2)\gamma_n k_1 \\ -\beta_\ell\varepsilon_n k_1 \end{array} &
\begin{array}{l} -\delta_\ell(\lambda_n+2)\gamma_n k_3 \\ -\beta_\ell\varepsilon_n k_3 \\ +\beta_\ell\xi_1 \end{array} &
\begin{array}{l} \alpha_\ell\lambda_n^2+(3+\nu_\ell)\alpha_\ell\lambda_n \\ +2(1+\nu_\ell)(1+\alpha_\ell)+(1+\nu_\ell)\lambda_n \\ +\varphi_\ell\lambda_n+2\varphi_\ell-\delta_\ell(\lambda_n+2)\gamma_n k_2 \\ -\beta_\ell\,\varepsilon_n k_2 \end{array} &
\begin{array}{l} -\alpha_\ell\lambda_n^2-(3+\nu_\ell)\alpha_\ell\lambda_n \\ -2(1+\nu_\ell)(1+\alpha_\ell)-\varphi_\ell\lambda_n \\ -2\varphi_\ell-\delta_\ell(\lambda_n+2)\gamma_n k_4 \\ -\beta_\ell\varepsilon_n k_4-\beta_\ell\xi_1 \end{array}
\end{bmatrix}
\begin{bmatrix} a_n \\ c_n \\ b_n \\ d_n \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\qquad (9.15)
$$

The case $n = 0$ for the system of equations (6.15) corresponds to constant values for all of the functions w_u, w_ℓ, ψ_u, ψ_ℓ. Hence it corresponds to constant values for w_u and w_ℓ and zero values for u_u and u_ℓ. This represents a uniform prebuckling state; since w_u, w_ℓ, ψ_u and ψ_ℓ represent buckling deformations from the prebuckling state, one may disregard the case $n = 0$ in the buckling analysis. Also, it will be shown below that λ_n is a common factor of all the terms in the buckling equation. Hence, for $n = 1$ (and thus $\lambda_n = 0$), <u>any</u> value of p may be inserted in the buckling equation, which vanishes due to the common factor of zero. Hence, for the buckling analysis one may divide out the common factor of λ_n and assume that $n \geq 2$.

For a given value of n, the system of equations (9.15) has nontrivial solutions if and only if the determinant of the coefficient matrix vanishes. This furnishes the *buckling equation* for each n.

It may be noted that equations (9.15) contain no approximations other than those inherent in the shell and core theories used. When the customary approximations (consistent with the Kirchhoff-Love shell theory) are made, the determinant of the coefficient matrix for equations (9.15) yields a <u>quadratic</u> equation for p. This is to be expected, since for each mode of buckling, there is a possible global buckling pressure and a (generally higher) ripple-type local buckling pressure. The price paid for this generality is the high order of the polynomial equation which yields stationary values of the buckling pressure p. In general, this is at least a twentieth-order polynomial equation. This general case is investigated in Section 13. Before proceeding to the general analysis however, two simpler but cruder analyses are presented (Sections 10 and 12). In the first special case, one makes the simplifying assumption $E_z = \infty$. This suppresses the face-wrinkling mode of deformation and hence yields a global buckling pressure. The reduction of the general formulation of equations (9.15) to the case of a monocoque spherical shell under external pressure is then conveniently made (Section 11). In the last case, the global buckling mode is suppressed and then the ripple buckling pressure may be derived.

The success of the simplified analyses of Sections 10 and 12 will, of course, depend on the effect of the simplifying assumptions made. The general analysis of Section 13 thus furnishes a basis for comparison.

10. Global Buckling with Rigid Core

Now assume that the modulus of elasticity E_z of the core is infinite. This eliminates the ripple-type deformation, but still allows the core to deform in shear. For this analysis, one must remove all terms involving the parameter E_z from the buckling determinant. This manipulation is shown below.

The buckling equation is

$$
\begin{vmatrix}
\begin{array}{l} -(1+\alpha_u)(\lambda_n+1+\nu_u) \\ +\varphi_u - \dfrac{\beta_u a\gamma_n k_1}{A_u} \end{array} &
\begin{array}{l} (1+\nu_u)(1+\alpha_u)+\alpha_u\lambda_n \\ -\varphi_u - \dfrac{\beta_u a\gamma_n k_3}{A_u} \end{array} &
-\dfrac{\beta_u a\gamma_n k_2}{A_u} &
-\dfrac{\beta_u a\gamma_n k_4}{A_u} \\
\begin{array}{l} \alpha_u\lambda_n^2 + (3+\nu_u)\alpha_u\lambda_n \\ +2(1+\nu_u)(1+\alpha_u) \\ +(1+\nu_u)\lambda_n + \varphi_u(\lambda_n+2) \\ -\delta_u(\lambda_n+2)\gamma_n k_1 + \dfrac{\beta_u p^*}{a_u} \\ -(\lambda_n+2) - \beta_u\varepsilon_n k_1 \end{array} &
\begin{array}{l} -\alpha_u\lambda_n^2 - (3+\nu_u)\alpha_u\lambda_n \\ -2(1+\nu_u)(1+\alpha_u) \\ -\varphi_u(\lambda_n+2) - \delta_u(\lambda_n+2)\gamma_n k_3 \\ -\dfrac{2\beta_u p^*}{a_u} - \beta_u\varepsilon_n k_3 - \beta_u\xi_1 \end{array} &
\begin{array}{l} -\delta_u(\lambda_n+2)\gamma_n k_2 \\ -\beta_u\varepsilon_n k_2 \end{array} &
\begin{array}{l} -\delta_u(\lambda_n+2)\gamma_n k_4 \\ -\beta_u\varepsilon_n k_4 + \beta_u\xi_1 \end{array} \\
\dfrac{a\beta_\ell\gamma_n k_1}{A_\ell} &
\dfrac{a\beta_\ell\gamma_n k_3}{A_\ell} &
\begin{array}{l} -(1+\alpha_\ell)(\lambda_n+1+\nu_\ell) \\ +\varphi_\ell + \dfrac{a\beta_\ell}{A_\ell}\gamma_n k_2 \end{array} &
\begin{array}{l} (1+\nu_\ell)(1+\alpha_\ell)+\alpha_\ell\lambda_n \\ -\varphi_\ell + \dfrac{a\beta_\ell\gamma_n}{A_\ell}k_4 \end{array} \\
\begin{array}{l} \delta_\ell(\lambda_n+2)\gamma_n k_1 \\ -\beta_\ell\varepsilon_n k_1 \end{array} &
\begin{array}{l} -\delta_\ell(\lambda_n+2)\gamma_n k_3 \\ -\beta_\ell\varepsilon_n k_3 + \beta_\ell\xi_1 \end{array} &
\begin{array}{l} \alpha_\ell\lambda_n^2 + (3+\nu_\ell)\alpha_\ell\lambda_n + 2(1+ \\ \nu_\ell)(1+\alpha_\ell) + (1+\nu_\ell)\lambda_n + \\ \varphi_\ell(\lambda_n+2) - \delta_\ell(\lambda_n+2) \\ \gamma_n k_2 - \beta_\ell\varepsilon_n k_2 \end{array} &
\begin{array}{l} -\alpha_\ell\lambda_n^2 - (3+\nu_\ell)\alpha_\ell\lambda_n \\ -2(1+\nu_\ell)(1+\alpha_\ell) \\ -\varphi_\ell(\lambda_n+2) - \delta_\ell(\lambda_n+2) \\ \gamma_n k_4 - \beta_\ell\varepsilon_n k_4 - \beta_\ell\xi_1 \end{array}
\end{vmatrix} = 0 \quad (10.1)
$$

After some manipulation, the buckling equation becomes

$$
\lambda_n \quad
\begin{vmatrix}
-1 & \begin{array}{l}(1+\nu_u)(1+\alpha_u)+\alpha_u\lambda_n \\ -\phi_u+\dfrac{\beta_u a\gamma_n k_1}{A_u}\end{array} & -\dfrac{\beta_u a\gamma_n k_2}{A_u} & -\dfrac{\beta_u a\gamma_n}{A_u}(k_2+k_4) \\[2ex]
\begin{array}{l}(1+\nu_u)+\dfrac{\beta_u}{\beta_\ell}(1+\nu_\ell) \\ +\dfrac{\beta_u}{a_u}p^*\end{array} & \begin{array}{l}-\alpha_u\lambda_n^2-(3+\nu_u)\alpha_u\lambda_n \\ -2(1+\nu_u)(1+\alpha_u)-\phi_u(\lambda_n+2) \\ +\dfrac{\beta_u}{\beta_\ell}(1+\nu_\ell)\lambda_n \\ +(\lambda_n+2)\gamma_n k_1(\delta_u+\dfrac{\beta_u}{\beta_\ell}\delta_\ell) \\ -\dfrac{2\beta_u}{a_u}p^*+2\beta_u\varepsilon_n k_1\end{array} & \begin{array}{l}\dfrac{\beta_u}{\beta_\ell}\{\alpha_\ell\lambda_n^2+(3+\nu_\ell)\alpha_\ell\lambda_n \\ +2(1+\nu_\ell)(1+\alpha_\ell)+(1+\nu_\ell)\lambda_n \\ +\phi_\ell(\lambda_n+2)\} \\ -(\lambda_n+2)\gamma_n k_2(\delta_u+\dfrac{\beta_u}{\beta_\ell}\delta_\ell) \\ -2\beta_u\varepsilon_n k_2\end{array} & \begin{array}{l}\dfrac{\beta_u}{\beta_\ell}(1+\nu_\ell)\lambda_n \\ -(\lambda_n+2)\gamma_n(k_2+k_4)(\delta_u+\dfrac{\beta_u}{\beta_\ell}\delta_\ell) \\ -2\beta_u\varepsilon_n(k_2+k_4)\end{array} \\[2ex]
-1 & -\lambda_n-\dfrac{a\beta_\ell\gamma_n k_1}{A_\ell} & \begin{array}{l}-(1+\alpha_\ell)(\lambda_n+1+\nu_\ell)+\phi_\ell \\ +a\dfrac{\beta_\ell\gamma_n k_2}{A_\ell}\end{array} & -\lambda_n-\dfrac{a\beta_\ell\gamma_n(k_2+k_4)}{A_\ell} \\[2ex]
(1+\nu_\ell) & \begin{array}{l}(1+\nu_\ell)\lambda_n+\delta_\ell(\lambda_n+2)\gamma_n k_1 \\ +\beta_\ell\varepsilon_n k_1\end{array} & \begin{array}{l}\alpha_\ell\lambda_n^2+(3+\nu_\ell)\alpha_\ell\lambda_n+2(1 \\ +\nu_\ell)(1+\alpha_\ell)+(1+\nu_\ell)\lambda_n \\ +\phi_\ell(\lambda_n+2)-\delta_\ell(\lambda_n+2)\cdot \\ \cdot\gamma_n k_2-\beta_\ell\varepsilon_n k_2\end{array} & \begin{array}{l}(1+\nu_\ell)\lambda_n-\delta_\ell(\lambda_n+2)\gamma_n\cdot \\ \cdot(k_2+k_4)-\beta_\ell\varepsilon_n(k_2+k_4) \\ -\beta_\ell\xi_1\end{array}
\end{vmatrix} = 0 \qquad (10.2)
$$

The relationship $k_1 + k_2 + k_3 + k_4 \equiv 0$ has been used in deriving equation (10.2). The term $[-\beta_\ell \xi_1]$ appearing in row four, column four of determinant (10.2) is the only one containing the parameter E_z. Factoring out this term and letting $E_z \rightarrow \infty$ while still requiring the resultant determinant to vanish yields the global buckling equation:

$$\begin{vmatrix} -1 & (1+\nu_u)(1+\alpha_u)+\alpha_u\lambda_n - \varphi_u + \dfrac{\beta_u a \gamma_n k_1}{A_u} & -\dfrac{\beta_u a \gamma_n k_2}{A_u} \\[2ex] \begin{matrix}(1+\nu_u) \\ + \dfrac{\beta_u}{\beta_\ell}(1+\nu_\ell) \\ + \dfrac{\beta_u}{a_u} p^*\end{matrix} & \begin{matrix} -\alpha_u\lambda_n^2 - (3+\nu_u)\alpha_u\lambda_n \\ -2(1+\nu_u)(1+\alpha_u) \\ -\varphi_u(\lambda_n+2) + \dfrac{\beta_u}{\beta_\ell}(1+\nu_\ell)\lambda_n \\ +(\lambda_n+2)\gamma_n k_1\left(\delta_u + \dfrac{\beta_u}{\beta_\ell}\delta_\ell\right) \\ -\dfrac{2\beta_u p^*}{a_u} + 2\beta_u \varepsilon_n k_1 \end{matrix} & \begin{matrix} \dfrac{\beta_u}{\beta_\ell}\Big\{\alpha_\ell\lambda_n^2 + (3+\nu_\ell)\alpha_\ell\lambda_n \\ + 2(1+\nu_\ell)(1+\alpha_\ell) + (1+\nu_\ell)\lambda_n \\ + \varphi_\ell(\lambda_n+2)\Big\} - (\lambda_n+2)\gamma_n k_2\Big(\delta_u \\ + \dfrac{\beta_u}{\beta_\ell}\delta_\ell\Big) - 2\beta_u \varepsilon_n k_2 \end{matrix} \\[2ex] -1 & -\lambda_n - \dfrac{a\beta_\ell\gamma_n k_1}{A_\ell} & \begin{matrix} -(1+\alpha_\ell)(\lambda_n+1+\nu_\ell) + \varphi_\ell \\ + \dfrac{a\beta_\ell\gamma_n k_2}{A_\ell}\end{matrix} \end{vmatrix} = 0 \quad (10.3)$$

or

$$
\begin{vmatrix}
-1 & (1+\nu_u)(1+\alpha_u)+\alpha_u\lambda_n-\varphi_u & -\dfrac{\beta_u a}{A_u} \\[2ex]
\begin{array}{l}
(1+\nu_u)+\dfrac{\beta_u}{\beta_\ell}(1+\nu_\ell) \\[1ex]
+\dfrac{\beta_u p^*}{a_u}+(\lambda_n+2)\Big\{\delta_u \\[1ex]
+\dfrac{\beta_u}{\beta_\ell}\delta_\ell+\dfrac{\beta_u h}{[1-(\frac{h}{2a})^2]}\Big\}\cdot \\[1ex]
\cdot\dfrac{A_u}{a\beta_u}
\end{array}
&
\begin{array}{l}
-\alpha_u\lambda_n^2-(3+\nu_u)\alpha_u\lambda_n \\[1ex]
-2(1+\nu_u)(1+\alpha_u) \\[1ex]
-\varphi_u(\lambda_n+2)+\dfrac{\beta_u}{\beta_\ell}\dfrac{k_1}{k_2}\cdot \\[1ex]
\cdot\Big\{\alpha_\ell\lambda_n^2+(3+\nu_\ell)\alpha_\ell\lambda_n \\[1ex]
+2(1+\nu_\ell)(1+\alpha_\ell) \\[1ex]
+(1+\nu_\ell)\lambda_n+\varphi_\ell(\lambda_n+2)\Big\} \\[1ex]
+\dfrac{\beta_u}{\beta_\ell}(1+\nu_\ell)\lambda_n-\dfrac{2\beta_u p^*}{a_u} \\[1ex]
-\Big\{(1+\nu_u)(1+\alpha_u)+\alpha_u\lambda_n \\[1ex]
-\varphi_u\Big\}\dfrac{A_u}{a\beta_u}(\lambda_n+2)\cdot \\[1ex]
\cdot\Big\{\delta_u+\dfrac{\beta_u}{\beta_\ell}\delta_\ell+\dfrac{h\beta_u}{[1-(\frac{h}{2a})^2]}\Big\}
\end{array}
&
\begin{array}{l}
\dfrac{\beta_u}{\gamma_n k_2\beta_\ell}\Big\{\alpha_\ell\lambda_n^2+(3+\nu_\ell)\alpha_\ell\lambda_n \\[1ex]
+2(1+\nu_\ell)(1+\alpha_\ell) \\[1ex]
+(1+\nu_\ell)\lambda_n+\varphi_\ell(\lambda_n+2)\Big\}
\end{array}
\\[2ex]
-\dfrac{\beta_\ell}{\beta_u}\dfrac{A_u}{A_\ell}-1
&
\begin{array}{l}
-\lambda_n-\dfrac{k_1}{k_2}(1+\alpha_\ell)(\lambda_n \\[1ex]
+1+\nu_\ell)+\dfrac{k_1}{k_2}\varphi_\ell \\[1ex]
+\dfrac{\beta_\ell}{\beta_u}\dfrac{A_u}{A_\ell}\big[(1+\nu_u)(1+\alpha_u) \\[1ex]
+\alpha_u\lambda_n\big]-\dfrac{\beta_\ell}{\beta_u}\dfrac{A_u}{A_\ell}\varphi_u
\end{array}
&
\begin{array}{l}
\big[-(1+\alpha_\ell)(\lambda_n+1+\nu_\ell) \\[1ex]
+\varphi_\ell\big]\dfrac{1}{\gamma_n k_2}
\end{array}
\end{vmatrix} = 0 \qquad (10.4)
$$

If one makes the usual approximations (consistent with the Kirchhoff-Love shell theory) of neglecting φ_u, φ_ℓ, α_u and α_ℓ compared to one, the buckling determinant (10.4) becomes

$$
\begin{vmatrix}
-1 & (1+\nu_u)+\alpha_u\lambda_n & -\dfrac{\beta_u a}{A_u} \\[2ex]
\begin{array}{l}
(1+\nu_u)+\dfrac{\beta_u}{\beta_\ell}(1+\nu_\ell) \\
+(\lambda_n+2)\left\{\delta_u+\dfrac{\beta_u}{\beta_\ell}\delta_\ell\right. \\
\left.+\dfrac{h\beta_u}{[1-(\frac{h}{2a})^2]}\;\dfrac{A_u}{a\beta_u}\right\}
\end{array}
&
\begin{array}{l}
-\alpha_u\lambda_n^2-(3+\nu_u)\alpha_u\lambda_n \\
-2(1+\nu_u)-\varphi_u(\lambda_n+2) \\
+\dfrac{\beta_u}{\beta_\ell}\dfrac{k_1}{k_2}\left\{\alpha_\ell\lambda_n^2+(3+\nu_\ell)\alpha_\ell\lambda_n\right. \\
\left.+(\lambda_n+2)(1+\nu_\ell)+\varphi_\ell(\lambda_n+2)\right\} \\
+\dfrac{\beta_u}{\beta_\ell}(1+\nu_\ell)\lambda_n-\left\{1+\nu_u\right. \\
\left.+\alpha_u\lambda_n\right\}\dfrac{A_u}{a\beta_u}(\lambda_n+2)\left\{\delta_u\right. \\
\left.+\dfrac{\beta_u}{\beta_\ell}\delta_\ell+\dfrac{h\beta_u}{[1-(\frac{h}{2a})^2]}\right\}
\end{array}
&
\begin{array}{l}
\dfrac{\beta_u}{\gamma_n k_2\beta_\ell}\left\{\alpha_\ell\lambda_n^2\right. \\
\left.+(\lambda_n+2)(1+\nu_\ell)\right\}
\end{array}
\\[2ex]
-\dfrac{\beta_\ell}{\beta_u}\dfrac{A_u}{A_\ell}-1
&
\begin{array}{l}
-\lambda_n-\dfrac{k_1}{k_2}(\lambda_n+1+\nu_\ell) \\
+\dfrac{\beta_\ell}{\beta_u}\dfrac{A_u}{A_\ell}[1+\nu_u+\alpha_u\lambda_n]
\end{array}
&
\dfrac{1}{\gamma_n k_2}[-(\lambda_n+1+\nu_\ell)]
\end{vmatrix} = 0 \quad (10.5)
$$

Equation (10.4) yields a <u>linear</u> equation in p, of the <u>form</u>

$$p = \frac{Q_o}{Q_1} = \frac{k_3\lambda_n^3 + k_2\lambda_n^2 + k_1\lambda_n + k_o}{\ell_3\lambda_n^3 + \ell_2\lambda_n^2 + \ell_1\lambda_n + \ell_o} \tag{10.6}$$

Hence the equation $\frac{dp}{d\lambda_n} = 0$ which yields the stationary values of p is the fourth-order equation

$$\begin{aligned}
&[k_3\ell_2 - \ell_3k_2]\ \lambda_n^4 + [2(k_3\ell_1 - \ell_3k_1)]\ \lambda_n^3 \\
&\quad + [3(k_3\ell_o - \ell_3k_o) + (k_2\ell_1 - \ell_2k_1)]\ \lambda_n^2 + [2(k_2\ell_o - \ell_2k_o)]\lambda_n \\
&\quad + [k_1\ell_o - \ell_1k_o] = 0
\end{aligned} \tag{10.7}$$

A computer program for finding p_{cr} from equations (10.6) and (10.7) is given in Appendix D. Graphical results for several particular sandwich shells are given in Chapter III.

11. Reduction to the Monocoque Shell

In order to reduce the formulation for a sandwich sphere to that for a monocoque shell, one must require that $E_z = \infty$, $G_z = \infty$, $h = 0$, $t_u = t_\ell = t$, $E_u = E_\ell = E$ and $\nu_u = \nu_\ell = \nu$. For this purpose, one may use equation (10.4), since it follows without approximation from (10.1) if $E_z = \infty$. Letting $G_z \to \infty$ and $h \to 0$ in equation (10.4), the buckling equation for the monocoque case becomes

$$\begin{vmatrix} \begin{array}{l} (1+\nu_u) + \dfrac{\beta_u}{\beta_\ell}(1+\nu_\ell) \\ + \dfrac{\beta_u p^*}{a_u} + (\lambda_n+2)\Big\{\delta_u \\ + \dfrac{\beta_u}{\beta_\ell}\delta_\ell\Big\} \dfrac{A_u}{a\beta_u} \end{array} & \begin{array}{l} -\alpha_u\lambda_n^2 - (3+\nu_u)\alpha_u\lambda_n - 2(1+\nu_u)(1+\alpha_u) \\ -\varphi_u(\lambda_n+2) + \dfrac{\beta_u}{\beta_\ell}\dfrac{k_1}{k_2}\Big\{\alpha_\ell\lambda_n^2 + (3+\nu_\ell)\alpha_\ell\lambda_n \\ +2(1+\nu_\ell)(1+\alpha_\ell) + (1+\nu_\ell)\lambda_n + \varphi_\ell(\lambda_n+2)\Big\} \\ + \dfrac{\beta_u}{\beta_\ell}(1+\nu_\ell)\lambda_n - \dfrac{2\beta_u p^*}{a_u} - \Big\{(1+\nu_u)(1+\alpha_u) \\ + \alpha_u\lambda_n - \varphi_u\Big\} \dfrac{A_u}{a\beta_u}(\lambda_n+2)\Big\{\delta_u + \dfrac{\beta_u}{\beta_\ell}\delta_\ell\Big\} \end{array} \\ \\ -\dfrac{\beta_\ell}{\beta_u}\dfrac{A_u}{A_\ell} - 1 & \begin{array}{l} -\lambda_n - \dfrac{k_1}{k_2}(1+\alpha_\ell)(\lambda_n+1+\nu_\ell) + \dfrac{k_1}{k_2}\varphi_\ell \\ + \dfrac{\beta_\ell}{\beta_u}\dfrac{A_u}{A_\ell}\left[(1+\nu_u)(1+\alpha_u)+\alpha_u\lambda_n\right] - \dfrac{\beta_\ell}{\beta_u}\dfrac{A_u}{A_\ell}\varphi_u \end{array} \end{vmatrix} = 0 \qquad (11.1)$$

Letting $\nu_u = \nu_\ell = \nu$, $E_u = E_\ell = E$, $t_u = t_\ell = t$, one gets

$$\frac{\beta_\ell}{\beta_u} = \frac{\beta_u}{\beta_\ell} = 1$$

$$\frac{k_1}{k_2} = -\frac{A_\ell}{A_u}$$

$$\frac{A_u}{a\beta_u}\left\{\delta_u + \frac{\beta_u}{\beta_\ell}\delta_\ell\right\} = \frac{t}{a_\ell}$$

$$N_u^o = -\frac{p^* a}{4A_u}$$

$$N_\ell^o = -\frac{p^* a}{4A_\ell}$$

$$A_u\varphi_u + A_\ell\varphi_\ell = -\frac{pa(1-\nu^2)}{2\,Et}$$

Now define

$$\alpha = \frac{A_u\alpha_u + A_\ell\alpha_\ell}{2} = \frac{t^2}{3a^2} = \frac{(2t)^2}{12a^2}$$

$$\varphi = -\frac{A_u\varphi_u + A_\ell\varphi_\ell}{2} = \frac{pa(1-\nu^2)}{2E(2t)}$$

The terms α and φ are the customary parameters of the monocoque analysis, but for a shell of total thickness $2t$.

Multiply column one of equation (11.1) by $\frac{A_\ell}{2}$ and column two by A_u, rearrange and neglect α and φ compared to one where necessary. Buckling equation (11.1) becomes

$$\begin{vmatrix} A_\ell(1+\nu) + \frac{t}{2a}\lambda_n & -2\alpha\lambda_n^2 - (3+\nu)2\alpha\lambda_n - 2(1+\nu)(2+2\alpha) + 2\varphi(\lambda_n+2) - \frac{t^2}{a^2A_\ell}(1+\nu)\lambda_n \\ -1 & -\frac{t}{a}\lambda_n + \frac{t^2}{6a^2A_\ell}\lambda_n + \frac{2+\frac{t^2}{6a^2}}{A_\ell}(1+\nu) \end{vmatrix} = 0 \qquad (11.2)$$

Expansion of equation (11.2) yields

$$\varphi = \frac{(1-\nu^2) + \alpha[\lambda_n^2 + 2\lambda_n + (1+\nu)^2]}{(\lambda_n+2)} \qquad (11.3)$$

This is exactly equation (A.13) of Appendix A, the buckling pressure as a function of λ_n, for a monocoque sphere of radius $2t$. Thus, the reduction of the general formulation to the monocoque case is complete.

12. Ripple Buckling

Ripple type buckling is defined as that buckling state characterized by opposing buckling mode shapes for the top and bottom sandwich facings. Thus, the buckling shapes top and bottom have different signs for their amplitudes. That ripple buckling involves the same mode $P_n[\cos\varphi]$ for both upper and lower facings may be verified by an inspection of equations (9.15), which shows that the only buckling modes possible for the spherical sandwich shell are those for which all of the displacement functions are nonzero multiples of $P_n[\cos\varphi]$. Global buckling, then, is that state in which the amplitudes of the corresponding displacement functions top and bottom are of the same sign (but not necessarily of the same numerical amplitudes); for ripple buckling, the deformation modes top and bottom have opposite signs (are opposed).

Thus, ripple buckling implies the existence of a surface in the core which undergoes no deformation during buckling. One may then analyze ripple buckling by solving two separate problems: buckling of the upper facing against that portion of the core above the "ripple interface," and buckling of the lower facing against that portion of the core below the "ripple interface." In this way, the ripple buckling pressure is completely separated from the global buckling pressure.

Since, however, this view of ripple buckling does not lead to any essential simplifications from the general formulation of Section 13, the problem of separating the ripple and global buckling is pursued there.

13. General Formulation

The general formulation involves the solution of the original system (9.15). If one neglects φ_u, φ_ℓ, α_u and α_ℓ compared to one, the determinantal equation for buckling becomes

$$
\begin{vmatrix}
\begin{array}{l} -(\lambda_n + 1 + \nu_u) \\ - \dfrac{\beta_u a \gamma_n k_1}{A_u} \end{array} &
\begin{array}{l} \alpha_u \lambda_n + 1 + \nu_u \\ - \dfrac{\beta_u a \gamma_n k_3}{A_u} \end{array} &
- \dfrac{\beta_u a \gamma_n k_2}{A_u} &
- \dfrac{\beta_u a \gamma_n k_4}{A_u} \\[2ex]
\begin{array}{l} \alpha_u \lambda_n^2 \\ +(\lambda_n+2)(1+\nu_u) \\ + \varphi_u(\lambda_n + 2) \\ -\delta_u(\lambda_n+2)\gamma_n k_1 \\ - \dfrac{\beta_u p^*}{a_u}(\lambda_n + 2) \\ -\beta_u \varepsilon_n k_1 \end{array} &
\begin{array}{l} -\alpha_u \lambda_n^2 - (3+\nu_u)\alpha_u \lambda_n \\ -2(1+\nu_u) - \varphi_u(\lambda_n+2) \\ -\delta_u(\lambda_n+2)\gamma_n k_3 \\ - \dfrac{2\beta_u p^*}{a_u} - \beta_u \varepsilon_n k_3 \\ - \beta_u \xi_1 \end{array} &
\begin{array}{l} -\delta_u(\lambda_n + 2)\gamma_n k_2 \\ -\beta_u \varepsilon_n k_2 \end{array} &
\begin{array}{l} -\delta_u(\lambda_n + 2)\gamma_n k_4 \\ -\beta_u \varepsilon_n k_4 + \beta_u \xi_1 \end{array} \\[2ex]
\dfrac{a\beta_\ell \gamma_n k_1}{A_\ell} &
\dfrac{a\beta_\ell \gamma_n k_3}{A_\ell} &
\begin{array}{l} -(\lambda_n + 1 + \nu_\ell) \\ + \dfrac{a\beta_\ell \gamma_n k_2}{A_\ell} \end{array} &
\begin{array}{l} \alpha_\ell \lambda_n + 1 + \nu_\ell \\ + \dfrac{a\beta_\ell \gamma_n k_4}{A_\ell} \end{array} \\[2ex]
\begin{array}{l} -\delta_\ell(\lambda_n+2)\gamma_n k_1 \\ -\beta_\ell \varepsilon_n k_1 \end{array} &
\begin{array}{l} -\delta_\ell(\lambda_n+2)\gamma_n k_3 \\ -\beta_\ell \varepsilon_n k_3 + \beta_\ell \xi_1 \end{array} &
\begin{array}{l} \alpha_\ell \lambda_n^2 + (\lambda_n+2)(1+\nu_\ell) \\ +\varphi_\ell(\lambda_n+2) - \delta_\ell(\lambda_n \\ + 2)\gamma_n k_2 - \beta_\ell \varepsilon_n k_2 \end{array} &
\begin{array}{l} -\alpha_\ell \lambda_n^2 - (3+\nu_\ell)\alpha_\ell \lambda_n - 2(1+\nu_\ell) \\ -\varphi_\ell(\lambda_n+2) - \delta_\ell(\lambda_n+2)\gamma_n k_4 \\ - \beta_\ell \varepsilon_n k_4 - \beta_\ell \xi_1 \end{array}
\end{vmatrix} = 0 \quad (13.1)
$$

The determinantal equation (13.1), when expanded, yields a <u>quadratic</u> equation in p:

$$Q_2 p^2 + Q_1 p + Q_o = 0 \qquad (13.2)$$

The coefficients Q_2, Q_1 and Q_o are, in turn, polynomials in λ_n:

$$Q_2 = AA\lambda_n^3 + BB\lambda_n^2 + CC\lambda_n + DD$$

$$Q_1 = PP\lambda_n^4 + QQ\lambda_n^3 + RR\lambda_n^2 + SS\lambda_n + TT$$

$$Q_o = FF\lambda_n^5 + GG\lambda_n^4 + HH\lambda_n^3 + JJ\lambda_n^2 + KK\lambda_n + LL$$

One may also write equation (13.2) as

$$\frac{Q_2}{Q_1}\ p^2 + p + \frac{Q_o}{Q_1} = 0 \qquad (13.3)$$

The solutions to equation (13.3) are

$$p_1 = \frac{1 + \sqrt{1 - \dfrac{4Q_oQ_2}{Q_1^2}}}{\dfrac{2Q_2}{Q_1}} \qquad (13.4)$$

$$p_2 = \frac{1 - \sqrt{1 - \dfrac{4Q_oQ_2}{Q_1^2}}}{\dfrac{2Q_2}{Q_1}} \qquad (13.5)$$

On physical grounds, it is conjectured that buckling does not change its <u>nature</u> as the material parameter E_z changes. Thus, it is conjectured that a global buckling pressure does not change into a ripple buckling pressure as E_z changes. Then, as $E_z \rightarrow \infty$, the ripple buckling pressure should also approach infinity, while the global buckling pressure remains finite. Now,

$$\lim_{E_z \to \infty} p_1 \text{ is infinite}$$

$$\lim_{E_z \to \infty} p_2 \text{ is finite.}$$

Hence, p_1 is the ripple buckling pressure and p_2 is the global buckling pressure.

$$p_{global} = \frac{1 - \sqrt{1 - \dfrac{4Q_oQ_2}{Q_1^{\,2}}}}{\dfrac{2Q_2}{Q_1}} \tag{13.6}$$

$$p_{ripple} = \frac{1 + \sqrt{1 - \dfrac{4Q_oQ_2}{Q_1^{\,2}}}}{\dfrac{2Q_2}{Q_1}} \tag{13.7}$$

The calculations for the stationary values of p_{global} and p_{ripple} are very lengthy; they are carried out in detail in Appendix E, where a computer program is given.

Literature Cited in Chapter II

1. Hildebrand, Reissner and Thomas," Notes on the Theory of Small Displacements of Orthotropic Shells," NACA Technical Note 1833, March 1949.

2. Love, A. E. H., A Treatise on the Mathematical Theory of Elasticity, Fourth Edition, Dover, New York, pp. 520-524; pp. 534-536.

CHAPTER III

COMPARISONS, CONCLUSIONS AND RECOMMENDATIONS

1. Comparisons

Figure 5 is a plot of buckling pressure versus core thickness for the simplified linear theory used in Chapter I. It is seen that the asymptotic buckling pressure p_{inf} which occurs for zero wavelength is always larger than (or possibly equal to) the critical pressure.

Figure 6 gives a comparison between the simplified linear theory and the general linear theory with $E_z = \infty$. It is seen that the two theories are quite close in their predictions of the critical pressure. It was to be expected on physical grounds that the general linear theory with rigid core would yield the higher values of the critical pressure since it includes the flexural rigidity of the facing, and a rigid core (while the simplified linear theory assumes membrane facings and an elastic core). This is confirmed in the figure.

Figure 7 gives a comparison among the simplified linear theory, the general linear theory with rigid core, and the general linear theory. Again on physical grounds, the simplest linear theory should give the least resistant shell, the general linear theory with rigid core should give the stiffest shell, and the general linear theory should lie in between. This is confirmed in the figure. It is seen also that all of the results are in close agreement with each other.

No experimental results on the stability of complete sandwich spheres have been found. Hence no comparison between theory and

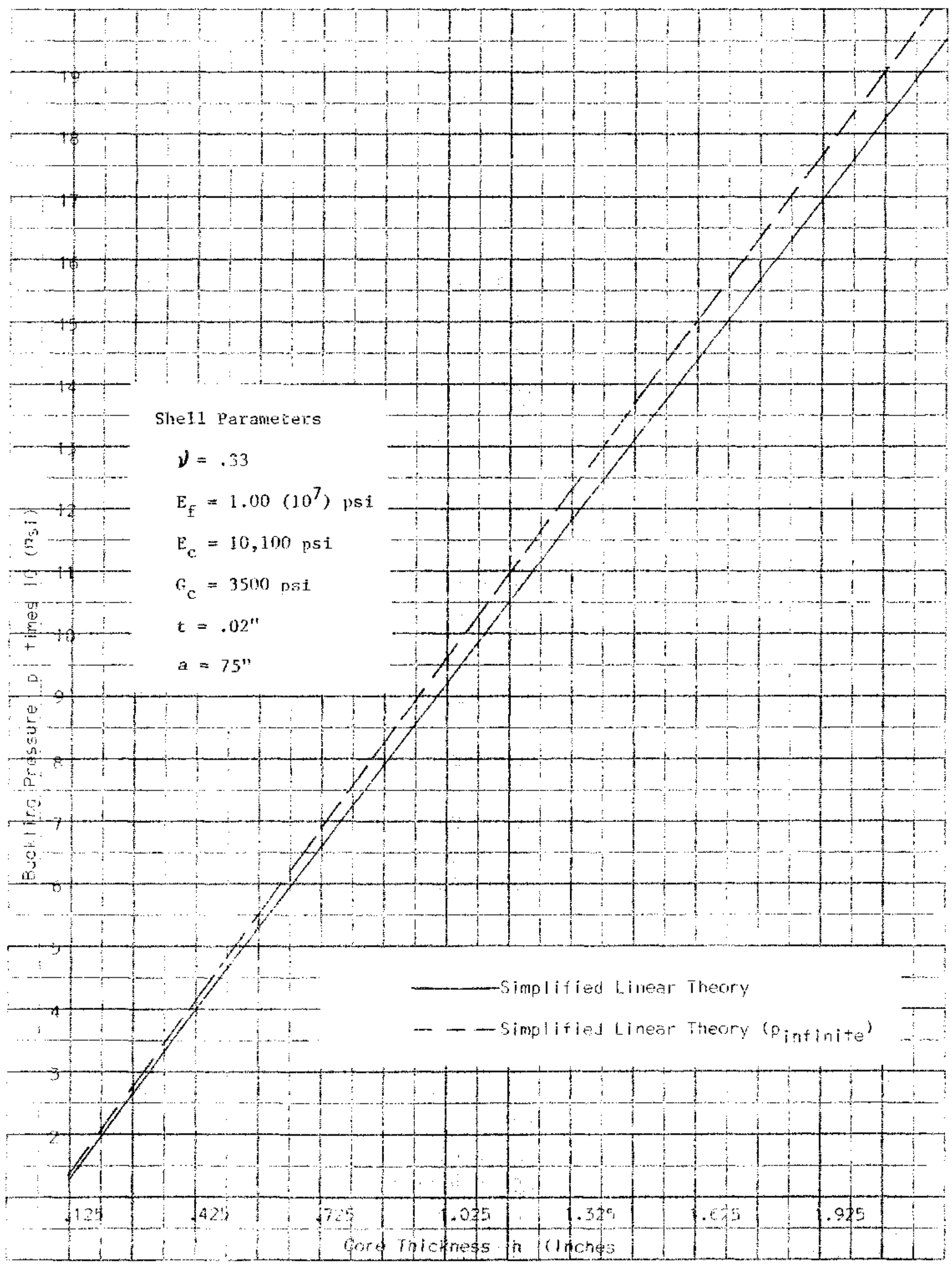

Figure 5. Comparison of Buckling Pressure with $p_{infinite}$ for Simplified Linear Theory.

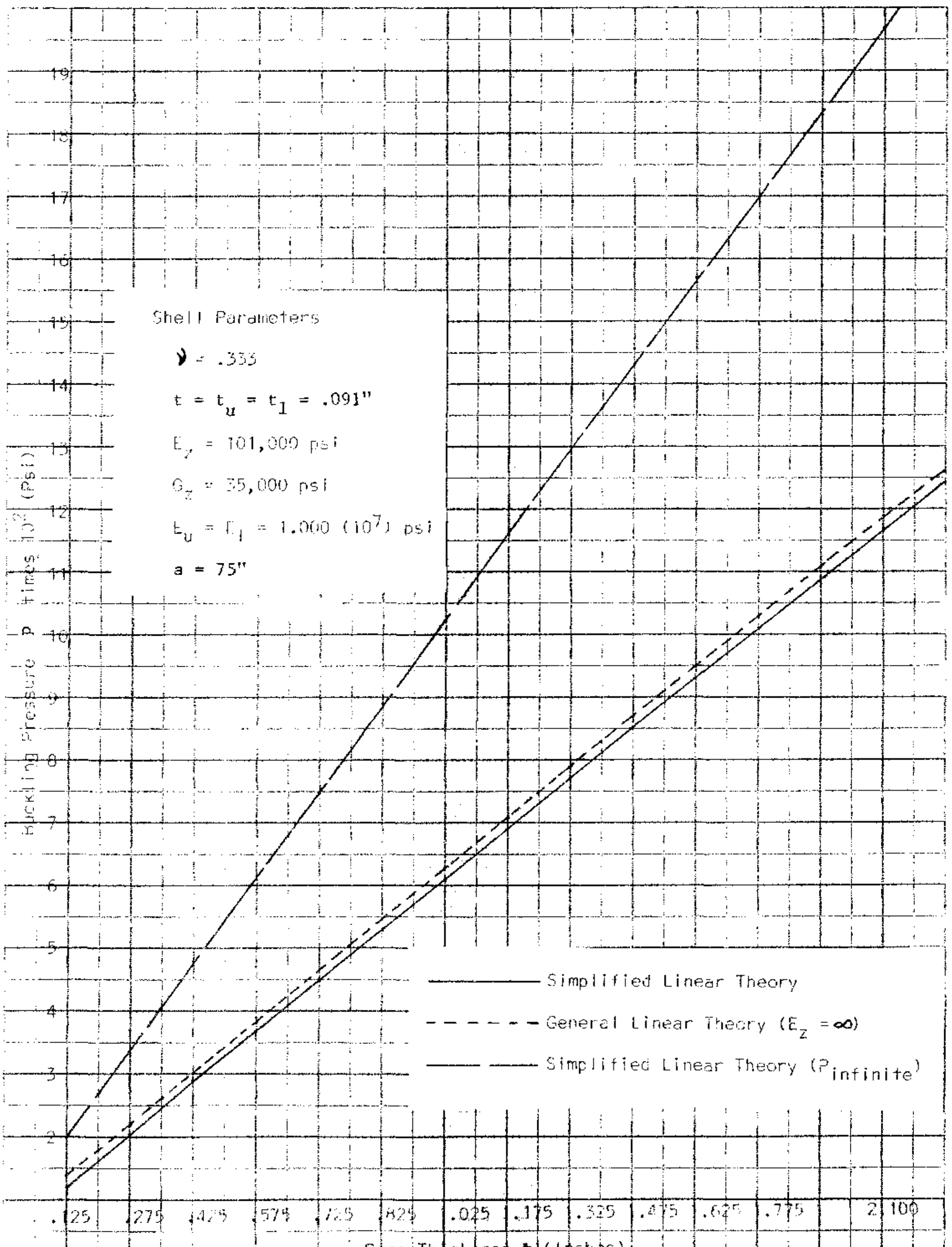

Figure 6. Comparison of Simplified Linear Theory and General Formulation with Rigid Core.

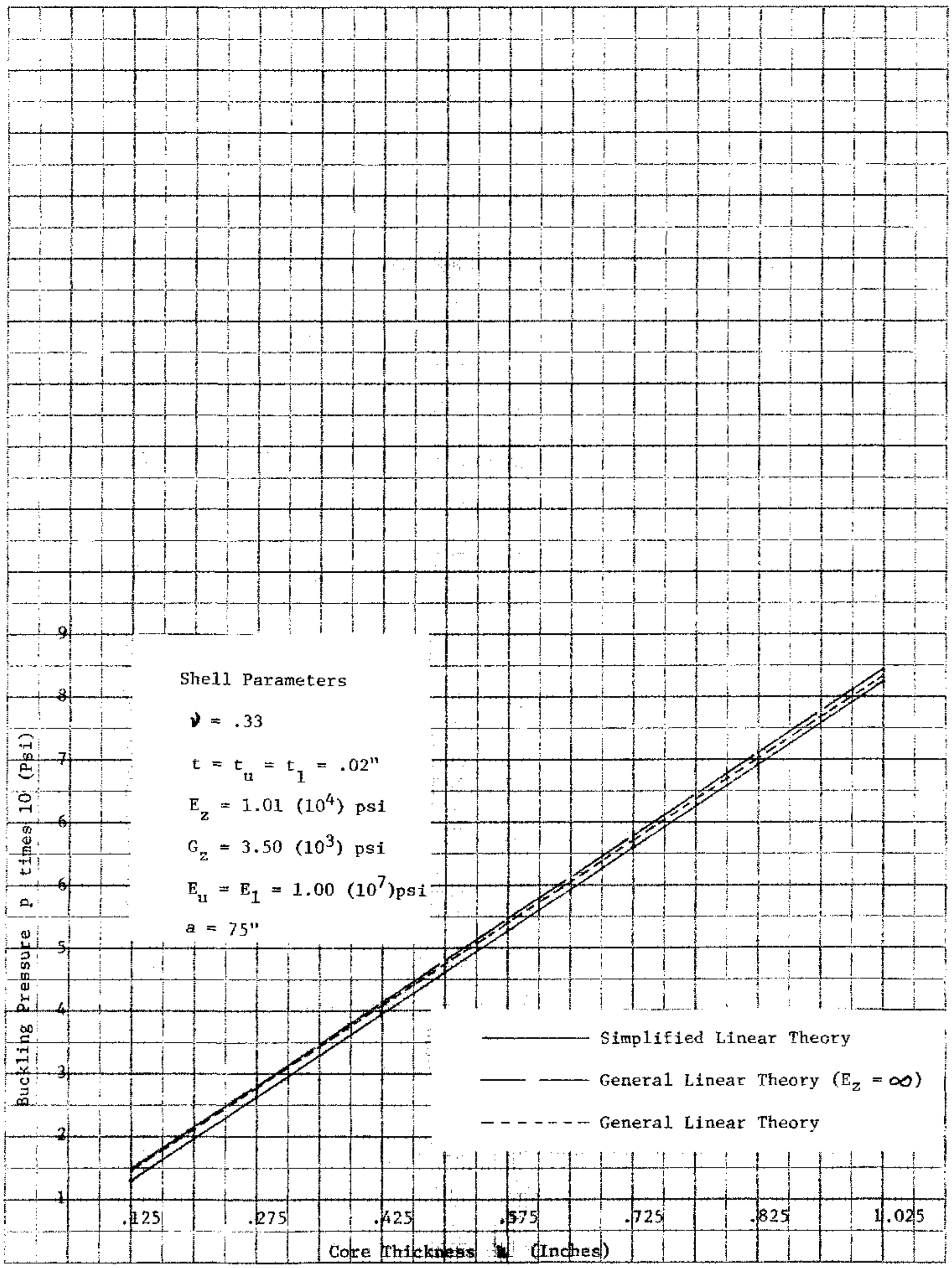

Figure 7. Comparison of Simplified Linear Theory and General Formulations.

experiment has been possible. In view of the well-known difficulties of similar experiments with monocoque spherical shells, this lack of experimental data is not surprising.

2. Conclusions

The problem of the buckling of a complete spherical sandwich shell under uniform external pressure has been solved using two different physical models. The first is Reissner's linear small deflection sandwich shell theory. It is shown that the buckling pressures may be "recovered" from a shell theory which does not include change of curvature terms if the proper buckling loadings are adopted (Appendix A). A quadratic equation for finding the critical pressure is found. In addition a third possibility, an asymptotic buckling pressure reached for zero buckling wavelength, is found. The critical pressure is then chosen from the three possibilities. The second model, a more general formulation, includes facings of nonzero flexural rigidities, different thicknesses and different elastic moduli, proper conditions of stress and displacement continuity at the sandwich interfaces, and a three-dimensional orthotropic elastic core in a state of antiplane stress. A simple solution is found for the global buckling pressure when the core is rigid in compression in the radial direction.

A goal of this thesis was the simple solution of the buckling problem, even for sandwich configurations whose top and bottom facings are unlike in both geometrical and elastic properties. It is felt that this goal has been attained with the inclusion of an efficient computer program for each analysis. With both models it is shown that the proper monocoque buckling pressure is attained when the sandwich shell is reduced to a monocoque one.

3. Recommendations

It is felt that further research might profitably be made along the following lines:

A. An analysis of the difference to be expected between a linear and a nonlinear formulation of the buckling problem for a spherical sandwich shell. This might be carried out by an analysis similar to that of Wang and Rao [3], but using a full circular ring as a model. Alternatively, an energy analysis yielding the exact slope of the *post-buckling* curve at bifurcation might be attempted. The above analyses, would, of course, include the effects of transverse shear in the sandwich core.

B. A nonlinear formulation using nonlinear theory for the facings but retaining the linearly elastic core. For this purpose, the analyses of Thompson [2] and Hyman [1] might be useful.

Literature Cited in Chapter III

1. Hyman, B. I., Buckling and Postbuckling Behavior of Prolate Spheroidal Shells under Uniform External Pressure, Ph.D. Thesis, Virginia Polytechnic Institute, Blacksburg, Virginia, September 1964.

2. Thompson, J. M. T., The Elastic Instability of Spherical Shells, Ph. D. Thesis, University of London, London, September 1961.

3. Wang, C. T. and Rao, G. V. R., "A Study of an Analogous Model Giving the Nonlinear Characteristics in the Buckling Theory of Sandwich Cylinders," Journal of the Aeronautical Sciences, Vol. 19, No. 1, January 1952, pp. 93-100.

APPENDIX A

DERIVATION OF CRITICAL BUCKLING PRESSURE USING LOVE'S SIMPLIFIED THEORY

Notation

$D = \dfrac{Eh^3}{12(1-\nu^2)}$	flexural rigidity
q	uniform external pressure
a	radius of sphere
h	shell thickness
x,y,z	orthogonal curvilinear shell coordinate in meridional, parallel circle, and inward radial directions, respectively
N_φ, N_θ	shell stress resultants in x and y directions, respectively
Q_φ	shear stress resultant
$N_o = \dfrac{-qa}{2}$	uniform prebuckling stress
u,v,w	deformations in x, y, and z directions, respectively
r_1, r_2	radii of curvature in meridional and parallel circle directions, respectively
φ	angle measured in meridional plane
$r_o = r_2 \sin\varphi$	radius of parallel circle
Y, Z	components of loading in y and z directions, respectively

Basic Differential Equations

$$\frac{d}{d\varphi}(N_\varphi r_o) - N_\theta r_1 \cos\varphi - r_o Q_\varphi + r_o r_1 Y = 0$$

$$N_\varphi r_o + N_\theta r_1 \sin\varphi + \frac{d}{d\varphi}(Q_\varphi r_o) + r_o r_1 Z = 0 \qquad \text{(A.1)}$$

$$\frac{d}{d\varphi}(M_\varphi r_o) - M_\theta r_1 \cos\varphi - r_o r_1 Q_\varphi = 0$$

For a spherical shell of radius a, and with $Y = 0$ and

$Z = -N_o\left(\frac{d}{ad\varphi} + \frac{\cot\varphi}{a}\right)\left(\frac{dw}{ad\varphi} + \frac{u}{a}\right)$, these equations become

$$\frac{dN_\varphi}{d\varphi} + \cot\varphi\,(N_\varphi - N_\theta)\,Q_\varphi = 0$$

$$N_\varphi + N_\theta + \frac{d}{d\varphi}Q_\varphi + Q_\varphi \cot\varphi - N_o\left(\frac{d}{d\varphi} + \cot\varphi\right)\left(\frac{dw}{ad\varphi} + \frac{u}{a}\right) = 0 \qquad \text{(A.2)}$$

$$\frac{dM_\varphi}{d\varphi} + \cot\varphi\,(M_\varphi - M_\theta) - aQ_\varphi = 0$$

The stress resultants N_x and N_y due only to the buckling deformations u, v and w are

$$N_x = C\left[\frac{du}{ad\varphi} - \frac{w}{a} + \nu\left(\frac{u\cot\varphi}{a} - \frac{w}{a}\right)\right]$$

$$N_y = C\left[\frac{u\cot\varphi}{a} - \frac{w}{a} + \nu\left(\frac{du}{ad\varphi} - \frac{w}{a}\right)\right]$$

$$\text{(A.3)}$$

$$M_x = \frac{-D}{a^2}\left[\frac{du}{d\varphi} + \frac{d^2w}{d\varphi^2} + \nu\left(u + \frac{dw}{d\varphi}\right)\cot\varphi\right]$$

$$M_y = \frac{-D}{a^2}\left[\left(u + \frac{dw}{d\varphi}\right)\cot\varphi + \nu\left(\frac{du}{d\varphi} + \frac{d^2w}{d\varphi^2}\right)\right]$$

where $C = \dfrac{Eh}{1-\nu^2}$ is the extensional rigidity of the material.

Let

$$\alpha = \frac{D}{a^2 C} = \frac{h^2}{12a^2}$$

$$\gamma = \frac{qa}{2C} = q\,\frac{a(1-\nu^2)}{2Eh} \tag{A.4}$$

Eliminating Q_φ from the first and last of equations (A.2) and substituting expressions (A.3) and (A.4) one obtains

$$(1+\alpha)\left[\frac{d^2u}{d\varphi^2} + \cot\varphi\,\frac{du}{d\varphi} - (\nu + \cot^2\varphi)u\right] - (1+\nu)\,\frac{dw}{d\varphi}$$

$$+\ \alpha\left[\frac{d^3w}{d\varphi^3} + \cot\varphi\,\frac{d^2w}{d\varphi^2} - (\nu + \cot^2\varphi)\,\frac{dw}{d\varphi}\right] = 0 \tag{A.5}$$

$$(1+\nu)\left(\frac{du}{d\varphi} + u\cot\varphi - 2w\right) + \alpha\left[\frac{-d^3u}{d\varphi^3} - 2\cot\varphi\,\frac{d^2u}{d\varphi^2}\right.$$

$$+\ (1+\nu+\cot^2\varphi)\,\frac{du}{d\varphi}$$

$$-\ \cot\varphi\,(2-\nu+\cot^2\varphi)u - \frac{d^4w}{d\varphi^4} - 2\cot\varphi\,\frac{d^3w}{d\varphi^3}$$

$$+\ (1+\nu+\cot^2\varphi)\,\frac{d^2w}{d\varphi^2}$$

$$\left.-\ \cot\varphi\,(2-\nu+\cot^2\varphi)\,\frac{dw}{d\varphi}\right] - \gamma\left(u\cot\varphi + \frac{du}{d\varphi}\right.$$

$$\left.+\ \cot\varphi\,\frac{dw}{d\varphi} + \frac{d^2w}{d\varphi^2}\right) = 0 \tag{A.6}$$

Let H denote the operator

$$H(\cdot) = \frac{d^2(\cdot)}{d\varphi^2} + \cot\varphi \frac{d(\cdot)}{d\varphi} + 2(\cdot)$$

Let $u = \frac{d\psi}{d\varphi}$.

Then, neglecting α in comparison with one, equations (A.5) and (A.6) become

$$H(\psi) + \alpha H(w) - (1 + \nu)(\psi + w) = 0 \tag{A.7}$$

$$\alpha HH(\psi + w) - (1 + \nu) H(\psi) - (3 + \nu) \alpha H(w) + 2(1 + \nu)(\psi + w)$$
$$+ \alpha[H(\psi) + H(w) - 2(\psi + w)] = 0 \tag{A.8}$$

Now, assume that

$$\psi = \sum_{n=o}^{\infty} A_n P_n (\cos\varphi) \quad \text{and}$$

$$w = \sum_{n=o}^{\infty} B_n P_n (\cos\varphi)$$

where $P_n(\cdot)$ is the Legendre function of order n.

Note that

$$H(P_n(\cos\varphi)) = -\lambda_n P_n$$

$$HH(P_n(\cos\varphi)) = \lambda_n^2 P_n$$

where $\lambda_n = n(n + 1) - 2$.

Then, equations (A.7) and (A.8) become

$$\sum_{n=o}^{\infty} [A_n[\lambda_n + (1 + \nu)] + B_n[\alpha\lambda_n + (1 + \nu)]] \cdot$$

$$\cdot P_n(\cos\varphi) = 0$$

$$\sum_{n=o}^{\infty} [A_n[\alpha\lambda_n^2 + (1 + \nu)(\lambda_n + 2) - \Upsilon(\lambda_n + 2)]$$

$$+ B_n[\alpha\lambda_n^2 + (3 + \nu)\alpha\lambda_n + 2(1 + \nu)$$

$$- \Upsilon(\lambda_n + 2)]] P_n(\cos\varphi) = 0 \qquad (A.9)$$

By the completeness of $\left\{P_n(\cos\varphi)\right\}_{n=o}^{\infty}$

$$A_n[\lambda_n + (1 + \nu)] + B_n[\alpha\lambda_n + (1 + \nu)] = 0 \qquad (A.10)$$

$$A_n[\alpha\lambda_n^2 + (1 + \nu)(\lambda_n + 2) - \Upsilon(\lambda_n + 2)]$$

$$+ B_n[\alpha\lambda_n^2 + (3 + \nu)\alpha\lambda_n + 2(1 + \nu) - \Upsilon(\lambda_n + 2)] = 0 \qquad (A.11)$$

This system of homogeneous equations in A_n and B_n has a nontrivial solution only if the determinant of the coefficient matrix vanishes. Hence, neglecting α and Υ in comparison with one, the requirement is that

$$(1 - \nu^2)\lambda_n + \alpha\lambda_n[\lambda_n^2 + 2\lambda_n + (1 + \nu)^2]$$

$$- \Upsilon(\lambda_n)(\lambda_n + 2) = 0 \qquad (A.12)$$

The solution $\lambda_n = 0$ of equation (A.12) corresponds to a translation of the shell and is disregarded for buckling. Hence,

$$(1 - \nu^2) + \alpha[\lambda_n{}^2 + 2\lambda_n + (1 + \nu)^2] - \Upsilon(\lambda_n + 2) = 0$$

or

$$\Upsilon = \frac{(1 - \nu^2) + \alpha[\lambda_n{}^2 + 2\lambda_n + (1 + \nu)^2]}{\lambda_n + 2} \tag{A.13}$$

Considering λ_n as a continuous rather than a discrete variable, extremizing Υ requires that

$$\frac{d\Upsilon}{d\lambda_n} = 0 \text{ ; or, neglecting small terms}$$

$$\lambda_n{}^2 + 4\lambda_n - \frac{1 - \nu^2}{\alpha} = 0$$

Thus $\lambda_n = -2 + \sqrt{1 + \frac{1 - \nu^2}{\alpha}}$

$$= -2 + \sqrt{\frac{1 - \nu^2}{\alpha}} \text{ , approximately.}$$

It is assumed that this value of λ_n corresponds to the minimum of Υ.

Then, from equation (A.13), $\Upsilon_{min} = 2\sqrt{(1 - \nu^2)\alpha} - 2\alpha$ or

$$q_{cr} = \frac{2Eh}{a(1 - \nu^2)}\left(\sqrt{\frac{1 - \nu^2}{3}}\ \frac{h}{a} - \frac{3h^2}{2a^2}\right) = \frac{C}{a}\ \Upsilon_{min}$$

Neglecting the last term, the following critical buckling pressure is obtained:

$$q_{cr} = \frac{2Eh}{a(1 - \nu^2)} \sqrt{\frac{1 - \nu^2}{3}}\ \frac{h}{a}$$

$$q_{cr} = \frac{2Eh^2}{a^2\sqrt{3(1 - \nu^2)}}$$

APPENDIX B

COMMENTS ON DR. YAO'S PAPER

The author has found only one previous investigation of the stability of a spherical sandwich shell; namely, the paper entitled "Buckling of Sandwich Sphere Under Normal Pressure," by John C. Yao (Ref. [9] in Chapter I). Several errors were noted; and some of the approximations made are not essential. These are discussed below.

1. In the section Buckling Mode, Dr. Yao states "The buckling mode for a clamped shallow monocoque sphere assumes concentric circular waves, which damp out at a certain distance from the center (Fig. 3). We assume the same mode for the sandwich sphere... Hence, at r = b,

$$\frac{d\omega}{a d\varphi} - \frac{u}{a} = \beta = 0$$

$$Q_{\Phi} \cos\varphi - N_{\varphi} \sin\varphi = 0 \text{ ."}$$

It should be noted that Dr. Yao's analysis is for a complete spherical shell, although this is not stated explicitly in his paper. This may be verified by considering his solution for the prebuckling stresses $N_{\Phi} = N_{\theta} = N_o = \frac{-pa}{2}$, which yields a uniform prebuckling radial deformation. The linear theory for a complete sphere predicts a waveform covering the whole sphere; Dr. Yao assumes that the sandwich sphere buckles in the well-known "dimple" of the nonlinear theory. Furthermore, a clamped shallow sphere experiences nonuniform deformation before buckling — a fact not predicted by the linear theory for the complete sphere. Dr. Yao required three boundary conditions at the first ridge of the dimple. That only two conditions need be imposed is shown below in (4).

2. Also in the section Buckling Mode, Dr. Yao reduces his equilibrium equations to shallow shell form. He writes "Equations (13), (14), (16), and (17) can, by use of equation (21), be rewritten as...." It may be noted, however, that his equation (22), which should follow from (13) when reduced to shallow shell form, does not contain the term in the shear Q_Φ. This omission of the shear term in the equilibrium equation is allowable in the Mushtari-Donnell simplification of the theory of shells, which the author later employs. However, the omission at this stage without explanation is incorrect.

3. In the section Stresses and Displacements, Dr. Yao says "Furthermore, terms involving N_o in equations (16) and (19), and terms involving M_o in equation (17), contribute nonlinear quantities which henceforth will be disregarded in the final expression." No derivation of these stress-displacement relations is given by Dr. Yao. The author believes that equations (26) of this analysis are in error. The correct expressions for N_Φ and N_θ contain terms in N_o which are linear in the displacements, and which are entirely omitted by Dr. Yao; furthermore, his expressions for M_Φ and M_θ involve a constant labeled by him as d_1, where

$$d_1 = \frac{D^*}{1 + 2(1 + \nu)\lambda - \nu^2}(1 + \lambda)$$

This constant should be $\frac{D^*(1 + \nu)}{1 + 2(1 + \nu)\lambda - \nu^2}$. For verification, one may refer to the author's present analysis in which these terms occur. These errors do not affect Dr. Yao's subsequent analysis, as he completely neglects all terms in M_o and N_o in his section Solution.

4. In his section Solution, Dr. Yao introduces the fundamental approximation in his analysis, the Mushtari-Donnell simplification of the theory of shells. The curvature terms are simplified and the equilibrium equation (22) omits the term in Q_Φ. In addition, Dr. Yao neglects the term in M_o which appears in equation (30) in his subsequent analysis. Furthermore, in his solution to the system of equations (28), (29), (30), he raises the order of the system. As is well known, this method may introduce extraneous constants in the solution to the original system. This has happened in Dr. Yao's calculations. The constant A_3 occurring in equation (40) is an extraneous constant whose value must be zero. This may be verified by substitution of Dr. Yao's equations (38), (39), and (40) into the original system (28), (29), and (30) (with the term in M_o omitted in (30)). At this point, the imposition of the three boundary conditions (20) may be considered. Only two constants A_1 and A_2 are available for their satisfaction. It is easily verified that the three boundary conditions are satisfied for non-trivial A_1 and A_2 only if $J_1(\frac{n_1 b}{a}) = J_1(\frac{n_2 b}{a}) = 0$. These are exactly Dr. Yao's equations (43) and (44). Thus, within the Mushtari-Donnell simplification, the boundary conditions (20), and the approximations made in the stress-displacement relations, the author agrees with Dr. Yao's solution (46), but not for his reasons. A rational basis for accepting his boundary conditions (20), at which he arrives by considering the nonlinear buckling mode, is as follows. One admits the possibility of a buckling waveform covering the sphere and searches for values $r = b$ such that the boundary conditions (20) are satisfied. The validity of this method will be established by

the existence of such values; and these have been found by Dr. Yao.

The major difference in the analysis of Chapter I and that of Dr. Yao is the absence in the present analysis of any approximations within the linear theory itself. Thus, the present analysis affords a check on the effect of the simplifications in the linear theory made by Dr. Yao. A comparison of critical pressures calculated for several spheres of different radii and material properties discloses only small differences. This indicates that these approximations are not critical as far as buckling pressures are concerned.

APPENDIX C

COMPUTER PROGRAM FOR THE SIMPLIFIED LINEAR THEORY

```
BEGIN
%        BUCKLING OF A SPHERICAL SANDWICH SHELL
%        THE SIMPLIFIED LINEAR THEORY   JOHN P ANDERSON
FILE IN          CARD (2,10) ;
FILE OUT         PRINT 4(3,15) ;
INTEGER          N,I ;
REAL                     LA,J,K,C1,C2,C3,C4,D1,D2,D3,D4,NO,K1,K2,KO,L2,L1,
         AA,BB,CC,DISCRIM,G,V,EF,EC,GC,A,H,T,ROOT ;
REAL ARRAY  LB,QODP,Q1DP,QO,Q1,P,PDP[O:3] ;
LABEL   FINISH,READCARD ;
FORMAT  HEADING(////,X10,"THE SIMPLIFIED LINEAR THEORY",/,
                     X10,"JOHN P ANDERSON",////,
                    X10,"V  IS POISSON S RATIO       ",/,
                    X10,"EF IS FACING MODULUS (PSI)",/,
                    X10,"EC IS CORE MODULUS (PSI)",/,
                    X10,"GC IS CORE RIGIDITY (PSI)",/,
                    X10,"A IS RADIUS (IN)",/,
                    X10,"H IS CORE THICKNESS (IN)",/,
                    X10,"T IS FACING THICKNESS (IN)",/,
                    X10,"P IS BUCKLING PRESSURE (PSI)",/,
                    X10,"LAMBA DEFINES THE BUCKLING LOAD") ;
FORMAT  HEADDATA(////,X10,"V",X9,"EF(PSI)",X9,"EC(PSI)",X9,"GC(PSI)",
        X9,"A(IN)",X9,"H(IN)",X9,"T(IN)") ;
FORMAT  FMI( /,X7,F5.4,X7,E9.2,X7 ,I6,X10,I6,X10,F6.2,X9,F5.3,X10,F5.3);
FORMAT  HEADRESULTS(//,X10,"P(PSI)",X20,"LAMBA",X26 ,"P DOUBLE PRIME") ;
FORMAT  RESULTS(   X10,E11.4,X13,F10.4,X13,F19.4) ;
FORMAT   RESULTS1(X10,E11.4,X45,F10.4) ;
LIST     LISTRESULTS1(P[3],PDP[3]) ;
FORMAT   FMT1(X36,"INFINITE ") ;
FORMAT  FMT3(X35,"IMAGINARY") ;
LIST    LISTRESULTS(P[I],LB[I],PDP[I]) ;
LIST     LSTIN(V,EF,EC,GC,A,H,T) ;
WRITE(PRINT,HEADING) ;
READCARD:        READ(CARD,/,LSTIN)[FINISH] ;
         FOR H ← .125 STEP .15 UNTIL 2.225 DO
         BEGIN
```

```
LA ← (H + T)×T×EF/(2×A*2×EC) ;
J  ← 2×T×EF/(A×(1 + 2×LA×(1+V)/3 - V*2)) ;
K  ← T×(H + T)*2×EF/(2×A×(1 + 2×LA×(1 + V) - V*2)) ;
C1 ← J×(1 + LA/3) ;
C2 ← J×(V -LA/3 ) ;
C3 ← J×(1 + V) ;
C4 ← -J×(1 + V)×(H + T)/(24×EC×A) ;
D1 ←  K×(1 + LA) ;
D2 ←  K×(V -LA) ;
D3 ← -((K×(1+V)×T×(H+T)×EF)/(4×A*3×(1+2×LA-V)×EC*2));
NO ←  - A×(1+(H+T)/(2×A))*2/2 ;
G ← (H + T)×GC ;
K2 ← - D1×C1×G/A ;
K1 ← - D1×C3×C1 + D1×C3×C2 + D1×C3×G/A + D1×C1×G/A
     - D2×C1×G/A ;
KO ← C3×(D1 - D2)×(C1 - C2) - A×G×C3×(C1 - C2) - C3×(D1 - D2)×G/A ;
%
L2 ← D1×C3×C4 + D1×C1×(-2×C4 + NO/A) - D3×C1×G ;
L1 ← (D1 - D2 - A×G)×(-C3×C4 - C1×(-2×C4 + NO/A)) + D3×C3×G ;
P[3] ← K2/L2 ;
%
AA ← K2×L1 - L2×K1 ;
BB ← - 2×L2×KO ;
CC ← - L1×KO ;
ROOT ← BB*2 -4×AA×CC ;
IF ROOT ≥ 0 THEN BEGIN
DISCRIM ← SQRT(BB*2 - 4×AA×CC) ;
LB[1]←(-BB + DISCRIM)/(2×AA) ;
LB[2]←(-BB - DISCRIM)/(2×AA) ;
PDP[3] ← 0 ;
FOR I ← 1 STEP 1 UNTIL 2 DO  BEGIN
QO[I]   ←  K2×LB[I]*2 + K1×LB[I] + KO ;
Q1[I]   ←  L2×LB[I]*2 + L1×LB[I] ;
QODP[I] ←  2×K2×LB[I] ;
Q1DP[I] ←  2×L2×LB[I] ;
PDP[I]  ←  QODP[I]/Q1[I] - Q1DP[I]×QO[I]/Q1[I]*2 ;
```

```
P[I]     ←  Q0[I]/Q1[I] ;
                                        END ;
%
WRITE(PRINT,HEADDATA) ;
WRITE(PRINT,FMT,LSTIN) ;
WRITE(PRINT,HEADRESULTS) ;
FOR I ← 1 STEP 1 UNTIL 2 DO
WRITE(PRINT,RESULTS,LISTRESULTS) ;
WRITE(PRINT[NO],RESULTS1,LISTRESULTS1) ;
WRITE(PRINT,FMT1) ;
                    END
ELSE
BEGIN
WRITE(PRINT,HEADDATA) ;
WRITE(PRINT,FMT,LSTIN) ;
WRITE(PRINT,HEADRESULTS) ;
WRITE(PRINT,FMT3) ;
WRITE(PRINT,FMT3) ;
WRITE(PRINT[NO],RESULTS1,LISTRESULTS1) ;
WRITE(PRINT,FMT1) ;
END ;
          END ;
              GO TO READCARD ;
FINISH ; END.
```

APPENDIX D

COMPUTER PROGRAM FOR THE GENERAL LINEAR THEORY WITH RIGID CORE (E_z IS INFINITE)

```
BEGIN
%        BUCKLING OF A  SPHERICAL SANDWICH SHELL    JOHN P ANDERSON
%        BEST LINEAR THEORY
%        GLOBAL BUCKLING  (EZ IS INFINITY)
$$ A A034                                                                  00000000
                                                                           99999900

$$ A A040                                                                  00000000
DVD: T ← (X2×X2 + Y2×Y2) ;                                                00007300
IF T = 0 THEN BEGIN X3 ← 1.0 ; Y3 ← 0 ; EXIT END ;                         00007400
                                                                           00007500
X3 ← SQRT(ABS((T+X1)/2)) ;                                                 00008600
Y3 ← (IF (T-X1) < 0 THEN 0 ELSE SQRT(ABS((T-X1)/2))) ;                     00008700
                                                                           99999900

%
FILE IN  CARD(2,10) ;
FILE OUT PRINT 4(2,15) ;
INTEGER  I      ;
REAL     A,H,TU,TL,EZ,GZ,EU,EL,VU,VL,M0,M1,M2,M3,N0,N1,N2,...,SAL,...,
         BU,BL,DU,DL,K1,K2,ALFU,ALFL,NU,P,NUL,P,PHIU,PHIL,...
         ,C21,A22,B22,C22,A23,B23,C23,C31,B32,C32,B33,C33,...,SUB,
         SUBONE,SUBTWO,GAMRK2 ;
REAL ARRAY    R,J,P,PCR,RC,IC,00,W1[0:4] ;
LABEL FINISH,READCARD,STEPAGAIN ;
LIST     LSTIN(A,H,TU,TL,GZ,EU,EL,VU,VL) ;
LIST     LISTRESULTS(R[I],J[I],P[I],PCR[I]) ;
FORMAT   HEADING(////,X10,"THE BEST LINEAR THEORY",/,
                     X10,"GLOBAL BUCKLING ( EZ IS INFINITY)",/,
                     X10,"JOHN P ANDERSON",/,
                     X10,"A IS RADIUS (IN)",/,
                     X10,"H IS CORE THICKNESS (IN)",/,
                     X10,"TU IS UPPER FACING THICKNESS (IN)",/,
                     X10,"TL IS LOWER FACING THICKNESS (IN)",/,
                     X10,"EZ IS CORE MODULUS (PSI)",/,
                     X10,"EU IS UPPER FACING MODULUS (PSI)",/,
                     X10,"EL IS LOWER FACING MODULUS (PSI)",/,
                     X10,"VU IS POISSON S RATIO, UPPER FACING",/,
```

```
                    X10,"VL IS POISSON S RATIO, LOWER FACING",/,
                    X10,"GZ IS CORE RIGIDITY (PSI)",/,
                    X10,"P IS BUCKLING PRESSURE (PSI)",/,
                    X10,"LAMBA DEFINES THE BUCKLING LOAD") ;
%
FORMAT    HEADDATA(////,X1 ,"A(IN)",X6,"H(IN)",X7,"TU(IN)",X7,"TL(IN)",
                X6,"GZ(PSI)",X8,"EU(PSI)",X8,"EL(PSI)",X9,"VU",X9,"VL"
                ) ;
FORMAT    FMT(/     ,F6.2,X6,F5.3,X7,F5.3,X8,F5.3,3(X6,E9.2),X6,F6.4,X5,
     F6.4) ;
FORMAT    HEADRESULTS(//,X22,"LAMBA",X31,"P(PSI)",X20,"P DOUBLE PRIME",/,
                  X10,"REAL PART",X13,"IMAG. PART",/,) ;
FORMAT    RESULTS(  X 8,E12.5,X11,E12.5,X12,E12.5,X18,R11.4) ;
FORMAT    IMAGFMT(  X8 ,E12.5,X11,E12.5) ;
FORMAT   FORMAT(X78,"THE ROOTS ARE IMAGINARY");
LIST      IMAGLIST(R[I],J[I]) ;
%
%
PROCEDURE       FUNCT(RZ,IZ,RVAL,IVAL) ;
VALUE           RZ,IZ ;
REAL            RZ,IZ,RVAL,IVAL ;
BEGIN
                ZPVAL( 4,RZ,IZ,RC,IC,RVAL,IVAL) ;
END FUNCT ;
          WRITE(PRINT,HEADING) ;
READCARD :     READ(CARD,/,LSTIN)[FINISH] ;
FOR H ← .125 STEP .15 UNTIL 2.225 DO BEGIN
AAU ← 1 + (H + TU)/(2×A) ;
AAL ← 1 - (H + TL)/(2×A) ;
AU  ← A×AAU ;
AL  ← A×AAL ;
BU ←(A*2)×(1 - VU*2)/(EU×TU) ;
BL ←(A*2)×(1 - VL*2)/(EL×TL) ;
DU ←(A*2)×(1 - VU*2)/(2×AAU×EU×(1 + H/(2×A)));
DL ←(A*2)×(1 - VL*2)/(2×AAL×EL×(1 - H/(2×A)));
K1DK2 ←-(H×TU/(4×AU) + A - H/2 - TU/(2×AAU))/(H×TL/(4×AL) + A + H/2
```

```
        + TL/(2×AAL)) ;
GAMRK2 ← - ((A*2)/GZ)×((3 + (H/(2×A))*2)/(3×(1 - (H/(2×A))*2)))×
            (H/((H×TL)/(4×AL) + A + H/2 + TL/(2×AAL))) ;
ALFU ←(TU*2)/(12×AU*2) ;
ALFL ←(TL*2)/(12×AL*2) ;
NOUP    ← - (A/(2×AAU)) × (1 - 1/((1 + H/(2×A))*2×(1 + ((1 - VL)/
          (1 - VU))×((EU×TU)/(EL×TL ))))) ;
NOLP    ← - (A/(2×AAL))×(1/((1 - H/(2×A))*2 ×
          (1 + ((1 - VL)/(1 - VU))×((EU × TU)/(EL×TL))))) ;
PHIU ← NOUP×(1 - VU*2)/(EU×TU) ;
PHIL ← NOLP×(1 - VL*2)/(EL×TL) ;
B12 ← ALFU ;
C12 ← 1 + VU ;
C13 ← - BU×A/AAU ;
SUB ← (DU + BU×DL/BL + H×BU/(1 - (H/(2×A))*2))×AAU/(A×BU) ;
B21 ← SUB ;
C21 ← 1 + VU + BU×(1 + VL)/BL + 2×SUB ;
C31 ← - BL×AAU/(BU×AAL) -1 ;
SUBONE ← BU×GAMRK2/BL ;
SUBTWO ← BU× K1DK2/BL ;
A22 ← - ALFU + SUBTWO×ALFL - ALFU×SUB  ;
B22 ← -(3 + VU)×ALFU + SUBTWO×(3 + VL)×ALFL
     + SUBTWO×(1 + VL) + BU×(1 + VL)/BL
     - SUB ×(  1 + VU    + 2×ALFU) ;
C22 ← -2×(1 + VU) + SUBTWO×2×(1 + VL) - SUB   ×(1 + VU)×2 ;
B ← - PHIU + SUBTWO×PHIL   ;
A23 ← SUBONE×ALFL ;
B23 ← SUBONE×(1 + VL) ;
C23 ← SUBONE×(1 + VL)×2 ;
B33 ←  - GAMRK2 ;
C33 ←  - GAMRK2×(1 + VL) ;
B32 ←  -1 - K1DK2  +(BL×AAU/(BU×AAL))×ALFU ;
C32 ←  -K1DK2×(1 + VL)   +   BL×AAU×(1 + VU)/(BU×AAL) ;
M3 ← -A22×B33 - B12×B21×B33 + B12×C31×A23 + A23×B32 ;
M2 ← -B22×B33 - A22×C33 + C13×B21×B32 - C31×A22×C13 + B23×B32 +
     A23×C32 - B12×(C21×B33 + B21×C33) + B12×C31×B23 - C12×B21×B33 +
```

```
    C12×C31×A23 ;
M1 ← -C22×B33 - B22×C33 + C13×(C21×B32 + B21×C32) + C23×B32 + B23×C32
     -C13×C31×B22 - B12×C21×C33 + B12×C31×C23  - C12×(C21×B33 + B21×C33)
     + C12×C31×B23 ;
M0 ← -C22×C33 + C13×C21×C32 - C13×C31×C22 + C23×C32 - C12×C21×C33 +
     C12×C31×C23 ;
%
%
N2 ← - B×B33 ;
N1 ← - 2×B×B33 - B×C33 - C31×C13×B ;
N0 ← - 2×B×C33 - 2×C13×C31×B ;
%
RC[4] ← M3×N2 ;
RC[3] ← 2×M3×N1 ;
RC[2] ← 3×(M3×N0) + M2×N1 - N2×M1 ;
RC[1] ← 2×(M2×N0 - N2×M0) ;
RC[0] ← M1×N0 - N1×M0 ;
FOR I ← 0 STEP 1 UNTIL 4 DO IC[I] ← 0 ;
FOR I ← 1 STEP 1 UNTIL 4 DO
BEGIN
RC[I] ← RC[I]/RC[0] ;
END ;
RC[0] ← 1 ;
WRITE    (PRINT,HEADDATA) ;
WRITE    (PRINT,FMT,LSTIN) ;
WRITE    (PRINT,HEADRESULTS) ;
         SOLVE(4,3,R,J,PRINT,FUNCT) ;
%
FOR I ← 1 STEP 1 UNTIL 4 DO BEGIN
         IF J[I] = 0
         THEN BEGIN
         Q0[I] ← M3×R[I]*3 + M2×R[I]*2 + M1×R[I] + M0 ;
         Q1[I] ←             N2×R[I]*2 + N1×R[I] + N0 ;
         P[I]  ←-Q0[I]/Q1[I] ;
         PDP[I] ←-(6×M3×R[I] + 2×M2 -  Q0[I]× 2×N2/Q1[I])/Q1[I] ;
         WRITE(PRINT,RESULTS,LISTRESULTS) ;
```

```
END
         ELSE
BEGIN
WRITE(PRINT[NO],IMAGFMT,IMAGLIST) ;
WRITE(PRINT,FORMAT) ;
END ;
         END ;
END ;
         GO TO READCARD ;
FINISH: END.
```

APPENDIX E

COMPUTER PROGRAM FOR THE GENERAL LINEAR THEORY

```
BEGIN
%        BUCKLING OF A SPHERICAL SANDWICH SHELL
%        BEST LINEAR THEORY  JOHN P ANDERSON
$$ A A034                                                             00000000
                                                                      99999999
$$ A A040                                                             00000000
DVD: T ← (X2×X2 +Y2×Y2) ;                                             00007000
IF T = 0 THEN BEGIN X3 ← 1.0 ; Y3 ← 0 ; EXIT END ;                    00007100
                                                                      00007200
X3 ← SQRT(ABS((T+X1)/2)) ;                                            00008300
Y3 ← (IF (T-X1) < 0 THEN 0 ELSE SQRT(ABS((T-X1)/2))) ;                00008400
                                                                      99999999
FILE IN        CARD (2,10) ;
FILE OUT       PRINT 4(3,15) ;
INTEGER        N,I ;
REAL ARRAY     Q0,Q1,Q2,Q0DP,Q1DP,Q2DP,PRIPPLE,PGLOBAL,PRIPPLEDP,J2,J1,
               PGLOBALDP,RC,IC,R,J,RCMOD[0:26] ;
LABEL          FINISH,READCARD  ;
REAL      AAU,AAL,AU,AL,K1,K2,K3,K4,GAM,X1,D1,D2,QUE,V0,NDU,NDL,
          ALFU,ALFL,PHIU,PHIL,CU,CL,FU,FL,C11,B11,C12,B12,N13,N14,
          A21,B21,C21,A22,B22,C22,B23,C23,N24,N31,N32,B33,C33,B34,
          C34,B41,C41,N42,A43,B43,C43,A44,B44,C44,BM2,BM3,BM5,CM2,
          CM3,CM5,NN24,NN42,A1,A2,A3,A4,B1,B2,B3,B4,XX0,XX1,XX2,XX3,
          XX4,XX5,RRR7,RR7,RA7,RB7,RC7,C1,C2,C3,C4,C5,C6,C7,C8,D1,
          D2,D3,D4,D5,D6,D7,D8,E1,E2,F3,E4,E5,E6,E7,E8,AA,BB,CC,DD,EE,
          RA1,RA2,RA3,RA4,RA5,RA6,RA8,RA9,RA10,RA11,RA12,RA13,RA15,
          RA17,RA18,RA19,RA21,RA22,RA23,RA24,RB1,RB2,RB3,RB4,RB5,RB6,
          RB8,RB9,RB10,RB11,RB12,RB13,RB15,RB17,RB18,RB19,RB20,RB21,
RB16,EEN,X1QUD,DU,DL,BU,BL,
          RB22,RB23,RB24,RC1,RC2,RC3,RC4,RC5,RC6,    RC8,RC9,RC10,RC11,
          RC12,RC13,RC14,RC15,RC16,RC17,RC18,RC19,RC20,RC21,RC22,RC23,
          RC24,RR5,RR6,RR9,RR10,RR11,RR12,RR15,RR17,RR18,RR19,RR24,
          RRR10,RRR15,RRR17,RRR19,RRR12,D15,E15,F15,G15,D14,E14,F14,
          G14,D13,E13,F13,G13,KA6,KB6,KC6,KD6,KA5,KB5,KC5,KD5,KA4,
          KB4,KC4,KD4,KE4,KF4,KG4,KH4,KA3,KB3,KC3,KD3,KE3,KF3,KG3,
          KH3,KA2,KB2,KC2,KD2,KE2,KF2,KG2,KH2,KA1,KB1,KC1,KD1,KE1,
```

```
         KF1,KG1,KH1,KA0,KB0,KC0,KD0,KE0,KF0,KG0,KH0,FF,GG,HH,JJ,
         KK,LL,NN,DZ1,DZ2,DZ3,DZ4,DZ5,DZ6,DZ7,DZ8,DZ9,DZ10,DZ11,
         DZ12,DZ13,DZ14,DZ15,DZ16,DZ17,DZ18,DZ19,DZ20,DZ21,DZ22,
         DZ23,DZ24,TA3,TB3,TC3,TD3,TE3,TF3,TG3,TH3,TA2,TB2,TC2,TD2,
         TE2,TF2,TG2,TH2,TA1,TB1,TC1,TD1,TE1,TF1,TG1,TH1,TA0,TB0,TC0,
         TD0,TE0,TF0,TG0,TH0,EZ1,EZ2,EZ3,EZ4,EZ5,EZ6,EZ7,EZ8,EZ9,EZ10,
         EZ11,EZ12,EZ13,EZ14,EZ15,EZ16,EZ17,EZ18,EZ19,EZ20,EZ21,EZ22,
         EZ23,EZ24,FZ1,FZ2,FZ3,FZ4,FZ5,FZ6,FZ7,FZ8,FZ9,FZ10,FZ11,FZ12,
         FZ13,FZ14,FZ15,FZ16,FZ17,FZ18,FZ19,FZ20,FZ21,FZ22,FZ23,FZ24,
         CZ2,CZ3,CZ4,CZ6,CZ8,CZ9,CZ10,CZ11,CZ12,CZ13,CZ15,CZ16,CZ17,
         CZ18,CZ19,CZ20,CZ21,CZ22,CZ23,CZ24,BZ17,BZ18,BZ19,BZ20,BZ21,
         BZ22,BZ23,BZ24,AZ17,AZ18,AZ19,AZ20,AZ21,AZ22,AZ23,AZ24,PP,QQ,
         RR,SS,TT,VV,A,H,TU,TL,EZ,GZ,EU,EL,VU,VL,Q110,Q19,Q18,Q17,Q16,
         Q15,Q14,Q13,Q12,Q11,Q10,S10,S9,S8,S7,S6,S5,S4,S3,S2,S1,S0,R10,
         R9,R8,R7,R6,R5,R4,R3,R2,R1,R0,T8,T7,T6,T5,T4,T3,T2,T1,T0,V8,
         V7,V6,V5,V4,V3,V2,V1,   VT8,VT7,VT6,VT5,VT4,VT3,VT2,VT1,VT0,
         Z16,Z15,Z14,Z13,Z12,Z11,Z10,Z9,Z8,Z7,Z6,Z5,Z4,Z3,Z2,Z1,Z0,
         PZ26,PZ25,PZ24,PZ23,PZ22,PZ21,PZ20,PZ19,PZ18,PZ17,PZ16,PZ15,
         PZ14,PZ13,PZ12,PZ11,PZ10,PZ9,PZ8,PZ7,PZ6,PZ5,PZ4,PZ3,PZ2,PZ1,
         PZ0,PX13,PX12,PX11,PX10,PX9,PX8,PX7,PX6,PX5,PX4,PX3,PX2,PX1,
         PX0,PY13,PY12,PY11,PY10,PY9,PY8,PY7,PY6,PY5,PY4,PY3,PY2,PY1,
         PY0,PXY13,PXY12,PXY11,PXY10,PXY9,PXY8,PXY7,PXY6,PXY5,PXY4,
         PXY3,PXY2,PXY1,PXY0,ZV26,ZV25,ZV24,ZV23,ZV22,ZV21,ZV20,ZV19,
         ZV18,ZV17,ZV16,ZV15,ZV14,ZV13,ZV12,ZV11,ZV10,ZV9,ZV8,ZV7,ZV6,
         ZV5,ZV4,ZV3,ZV2,ZV1,ZV0 ; % FINISH OF REAL
LIST     LSTIN(A,H,TU,TL,EZ,GZ,EU,EL,VU,VL) ;
%
%
FORMAT   HEADDATA(////,"A(IN)",X6,"H(IN)",X6,"TU(IN)",X6, "TL(IN)",
         X6,"EZ(PSI)",X6,"GZ(PSI)",X6,"EU(PSI)", X8,"EL(PSI)", X8,
         "VU",X8,"VL" ) ;
FORMAT   FMT(/,F6.2,X5,F5.3,X6 ,F5.3,X7,F5.3, 4(X5,E9.2),X6,F6.4,X6,
         F6.4) ;
%
FORMAT  HEADRESULTS(//,"ROOT NO.",X9 ,"LAMBA",X18,"P GLOBAL(PSI)",/,X10,
       "REAL PART",X3,"IMAG. PART") ;
```

```
%
LIST     LISTRESULTS(I , R[I] , J[I] , PGLOBAL[I]) ;
%
FORMAT  FOMMAT(X40,"THE ROOTS ARE IMAGINARY");
FORMAT   RESULTS( X3,I2,X4, E10.3,X2, E10.3 , X9, E12.5) ;
FORMAT   IMAGFMT    (X3,I2,X4,E10.3,X2,E10.3   ) ;
LIST     IMAGLIST(I , R[I] , J[I] ) ;
PROCEDURE      FUNCT(RZ,IZ,RVAL,IVAL) ;
VALUE          RZ,IZ ;
REAL           RZ,IZ,RVAL,IVAL ;
BEGIN
               ZPVAL( 4,RZ,IZ,RC,IC,RVAL,IVAL) ;
END FUNCT ;
READCARD:      READ(CARD,/,LSTIN)[FINISH] ;
FOR H ← .125 STEP .15 UNTIL 2.225 DO BEGIN
WRITE(PRINT,HEADDATA) ;
WRITE(PRINT,FMT,LSTIN) ;
WRITE(PRINT,HEADRESULTS) ;
AAU ← 1 + (H + TU)/(2×A) ;
AAL ← 1 - (H + TL)/(2×A) ;
AU  ← A×AAU ;
AL  ← A×AAL ;
BU ←  A*2×(1 - VU*2)/(EU×TU) ;
BL ←  A*2×(1 - VL*2)/(EL×TL) ;
DU ←  A*2×(1 - VU*2)/(2×AAU×EU×(1 + H/(2×A))) ;
DL ←  A*2×(1 - VL*2)/(2×AAL×EL×(1 - H/(2×A))) ;
K1 ←  TU/(4×AU) + A/H - 1/2 - TU/(2×H×AAU) ;
K2 ← - ( TL/(4×AL) + A/H + 1/2 + TL/(2×H×AAL)) ;
K3 ← 1/2 - TU/(4×AU) + TU/(2×H×AAU) ;
K4 ← 1/2 + TL/(4×AL) + TL/(2×H×AAL) ;
GAM ←  GZ/A*2 ;
EEN ← H×GZ/(2×A*2) ;
X1 ← EZ/H ;
X1QUO ← A*2×(1 - VU)/( 2×EU×TU×( 1/X1 + A*2×(1 - VU)/(2×EU×TU)
          + A*2×(1 - VL)/(2×EL×TL))) ;
NOU ← - A×( 1 - X1QUO/(1 + H/(2×A))*2) /(2×AAU) ;
```

```
NOL ← - A×( X1QUO/( 1 - H/(2×A))*2)/(2×AAL) ;
% ADD EEN AND X1QUO TO REAL  ALSO DU DL
PHIU ← NOU×(1 - VU*2)/(EU×TU) ;
PHIL ← NOL×(1 - VL*2)/(EL×TL) ;
ALFU ← TU*2/(12×AU*2) ;
ALFL ← TL*2/(12×AL*2) ;
B11 ← -1 ;
B12 ← ALFU ;
B21 ← 1 + VU - DU×GAM×K1 - BU×EEN×K1 ;
B22 ← -(3 + VU)×ALFU - DU×GAM×K3 - BU×EEN×K3 ;
B23 ← -DU×GAM×K2 - BU×EEN×K2 ;
B33 ← -1 ;
B34 ←-ALFL ;
B41 ← - DL×GAM×K1 - BL×EEN×K1 ;
B43 ← 1 + VL - DL×GAM×K2 - BL×EEN×K2 ;
B44 ← +(3 + VL)×ALFL + DL×GAM×K4 + BL×EEN×K4 ;
%
%
A21 ← ALFU ;
A22 ← -ALFU ;
A43 ← ALFL ;
A44 ←  ALFL ;
%
%
C11 ← -(1 + VU + BU×A×GAM×K1/AAU) ;
C12 ←   1 + VU - BU×A×GAM×K3/AAU  ;
C21 ← 2×(1 + VU) - 2×BU×EEN×K1 - 2×DU×GAM×K1 ;
C22 ←-2×(1 + VU) - 2×DU×GAM×K3 - 2×BU×EEN×K3 - BU×X1 ;
C33 ← -(1 + VL) + A×BL×GAM×K2/AAL ;
C34 ←-(1 + VL + A×BL×GAM×K4/AAL);
C41 ← -2×DL×GAM×K1 - 2×BL×EEN×K1 ;
C43 ← 2×(1 + VL)-2×DL×GAM×K2- 2×BL×EEN×K2 ;
C44 ←  2×(1 + VL) +2×DU×GAM×K4 + 2×BL×EEN×K4 + BL×X1 ;
C23 ← -2×DU×GAM×K2 - 2×BU×EEN×K2 ;
%
N13 ← - BU×A×GAM×K2/AAU ;
```

```
N14 ←   BU×A×GAM×K4/AAU ;
NN24 ←   DU×GAM×K4 + BU×EEN×K4 ;
N24  ←   2×DU×GAM×K4 + 2×BU×EEN×K4 − BU×X1 ;
N31 ← A×BL×GAM×K1/AAL ;
N32 ← A×BL×GAM×K3/AAL ;
NN42 ← − DL×GAM×K3 − BL×EEN×K3 ;
N42 ← − 2×DL×GAM×K3 − 2×BL×EEN×K3 + BL×X1 ;
%
%
BM2 ← PHIU + BU/AU ;
BM3 ← − PHIU ;
BM5 ← PHIL ;
%
%
CM2 ← 2×PHIU + 2×BU/AU ;
CM3 ← −2×PHIU − 2×BU/AU ;
CM5 ← 2×PHIL ;
A1 ←  B33×BM3×BM5×B11 ;
A2 ← −B11×B34×BM3×BM5 ;
A3 ← −BM2×BM5×B12×B33 ;
A4 ←  BM2×B34×BM5×B12 ;
%
B1 ←  B33×BM3×(CM5×B11 + BM5×C11) + BM5×B11×(C33×BM3 + B33×CM3) ;
B2 ← −B11×B34×(CM3×BM5 + BM3×CM5) − BM3×BM5×(C11×B34 + B11×C34) ;
B3 ← −BM2×BM5×(C12×B33 + B12×C33) − B12×B33×(CM2×BM5 + BM2×CM5) ;
B4 ←  BM2×B34×(CM5×B12 + BM5×C12) + BM5×B12×(CM2×B34 + BM2×C34) ;
%
C1 ←  B33×BM3×CM5×C11 + (C33×BM3 + B33×CM3)×(CM5×B11 + BM5×C11)
      +C33×CM3×BM5×B11  ;
C2 ← −B11×B34×CM3×CM5 − (C11×B34 + B11×C34)×(CM3×BM5 + BM3×CM5)
      −C11×C34×BM3×BM5  ;
C3 ← −BM2×BM5×C12×C33 − (CM2×BM5 + BM2×CM5)×(C12×B33 + B12×C33)
      −CM2×CM5×B12×B33  ;
C4 ←  BM2×B34×CM5×C12 + (CM2×B34 + BM2×C34)×(CM5×B12 + BM5×C12)
      +CM2×C34×BM5×B12  ;
XX0 ←  NN24×NN42 ;
```

```
XX1 ←  N24×NN42 + N42×NN24 ;
XX2 ←  N24×N42 ;
XX3 ←  B11×C33 + B33×C11 ;
XX4 ←  NN24/N24 ;
XX5 ←  NN42/N42 ;
RRR7 ←  XX0×B11×B33 ;
RR7  ←  XX1×B11×B33 + XX3×XX0 ;
RA7  ←  XX2×B11×B33 + XX3×XX1 + C11×C33×XX0 ;
RB7  ←  XX3×XX2 + C11×C33×XX1 ;
RC7  ←  C11×C33×XX2 ;
C5 ← -N14×N32×BM2×BM5  ;
C6 ← +N13×N32×BM2×BM5  ;
C7 ←  N31×N14×BM3×BM5  ;
C8 ← -N31×N13×BM3×BM5  ;
%
%
D1 ←  CM5×C11×(C33×BM3 + B33×CM3) + C33×CM3×(CM5×B11 + BM5×C11) ;
D2 ← -CM3×CM5×(C11×B34 + B11×C34) - C11×C34×(CM3×BM5 + BM3×CM5) ;
D3 ← -C33×C12×(CM2×BM5 + BM2×CM5) - CM2×CM5×(C12×B33 + B12×C33) ;
D4 ←  CM5×C12×(CM2×B34 + BM2×C34) + CM2×C34×(CM5×B12 + BM5×C12) ;
D5 ← -N14×N32×(BM2×CM5 + CM2×BM5) ;
D6 ←  N13×N32×(BM2×CM5 + BM5×CM2) ;
D7 ←  N31×N14×(BM3×CM5 + BM5×CM3) ;
D8 ← -N31×N13×(BM3×CM5 + BM5×CM3) ;
%
%
E1 ←  C33×CM3×CM5×C11 ;
E2 ← -C11×C34×CM3×CM5 ;
E3 ← -CM2×CM5×C33×C12 ;
E4 ←  CM2×C34×CM5×C12 ;
E5 ← -N14×N32×CM2×CM5 ;
E6 ←  N13×N32×CM2×CM5 ;
E7 ←  N31×N14×CM3×CM5 ;
E8 ← -N31×N13×CM3×CM5 ;
%
%
```

```
AA ← A1 + A2 + A3 + A4 ;
BB ← B1 + B2 + B3 + B4 ;
CC ← C1 + C2 + C3 + C4 + C5 + C6 + C7 + C8 ;
DD ← D1 + D2 + D3 + D4 + D5 + D6 + D7 + D8 ;
EE ← E1 + E2 + E3 + E4 + E5 + E6 + E7 + E8 ;
%
BEGIN  REAL  SHELL  ;
%
RA1 ←  B11×B33;
RA2 ←  B11×B34 ;
RA3 ←  B12×B33 ;
RA4 ←  B34×B12 ;
RA5 ←  N42×(B11×(C23×B34 + B23×C34) + C11×B23×B34) ;
RA6 ←  N24×N32×(B11×B43 + C11×A43) ;
RA8  ← B11×B23 ;
RA9  ← N13×N42×(B21×B34 + A21×C34) ;
RA10 ← N14×N32×(A21×C43 + B21×B43 + C21×A43) ;
RA11 ← N14×N42×(A21×C33 + B21×B33) ;
RA12 ← N13×N32×(A21×C44 + B21×B44 + A44×C21) ;
RA13 ← N31×B23×B12 ;
RA15 ← N31×N14×(A43×C22 + B43×B22 + C43×A22) ;
RA17 ← N31×N13×(A22×C44 + B22×B44 + A44×C22) ;
RA18 ← N31×N24×(B12×B43 + C12×A43) ;
RA19 ← B41×B23×C12×C34 + (C41×B23 + B41×C23)×(C12×B34 + B12×C34) +
       C41×C23×B12×B34 ;
RA21 ← N14×B41×B33 ;
RA22 ← N14×N32×B41×B23 ;
RA23 ← N13×B41×B34 ;
RA24 ← N24×(C41×B12×B33 + B41×(C12×B33 + B12×C33)) ;
%
%
RB1 ← B11×C33 + B33×C11 ;
RB2 ← B11×C34 + B34×C11 ;
RB3 ← B12×C33 + B33×C12 ;
RB4 ← B12×C34 + B34×C12 ;
RB5 ← N42×(B11×C23×C34 + C11×(C23×B34 + B23×C34)) ;
```

```
RB6 ← N24×N32×(B11×C43 + B43×C11) ;
RB8 ← B23×C11 + B11×C23 ;
RB9 ← N13×N42×(B34×C21 + B21×C34) ;
RB10 ←N14×N32×(B21×C43 + C21×B43) ;
RB11 ←N14×N42×(B21×C33 + C21×B33) ;
RB12 ←N13×N32×(B21×C44 + C21×B44) ;
RB13 ←     N31×(B23×C12 + C23×B12) ;
RB15 ←N31×N14×(B43×C22 + C43×B22) ;
RB16 ← N31×N14×N42×B23 ;
RB17 ← N31×N13×(B22×C44 + C22×B44) ;
RB18 ← N31×N24×(B12×C43 + C12×B43) ;
RB19 ← C12×C34×(C41×B23 + B41×C21) + C41×C23×(C12×B34 + B12×C34) ;
RB20 ← B41×N13×N24×N32 ;
RB21 ← N14×(B41×C33 + C41×B33) ;
RB22 ← N14×N32×(B41×C23 + C41×B23) ;
RB23 ← N13×(B41×C34 + C41×B34) ;
RB24 ← N24×(B41×C12×C33 + C41×(C12×B33 + B12×C33)) ;
%
%
RC1 ← C11×C33 ;
RC2 ← C11×C34 ;
RC3 ← C12×C33 ;
RC4 ← C34×C12 ;
RC5 ← C11×C23×C34×N42 ;
RC6 ← N24×N32×C11×C43 ;
RC8 ← C11×C23 ;
RC9 ← N13×N42×C21×C34 ;
RC10 ←N14×N32×C21×C43 ;
RC11 ← N14×N42×C21×C33 ;
RC12 ← N13×N32×C21×C44 ;
RC13 ← N31×C23×C12 ;
RC14 ← N31×N13×N24×N42 ;
RC15 ← N31×N14×C43×C22 ;
RC16 ← N31×N14×C23×N42 ;
RC17 ← N31×N13×C22×C44 ;
RC18 ← N31×N24×C12×C43 ;
```

```
RC19 ← C41×C23×C12×C34 ;
RC20 ← N13×N24×N32×C41 ;
RC21 ← N14×C41×C33     ;
RC22 ← N14×N32×C41×C23 ;
RC23 ← C41×N13×C34     ;
RC24 ← N24×C41×C12×C33 ;
%
%
RR5 ← N42×B11×B23×B34 ;
RR6 ← N24×N32×B11×A43 ;
RR9 ← N13×N42×A21×B34 ;
RR10 ← N14×N32×(A21×B43 + B21×A43) ;
RR11 ← N14×N42×A21×B33 ;
RR12 ← N13×N32×(A21×B44 + B21×A44) ;
RR15 ← N31×N14×(A43×B22 + B43×A22) ;
RR17 ← N31×N13×(A22×B44 + B22×A44) ;
RR18 ← N31×N24×B12×A43 ;
RR19 ← B41×B23×(C12×B34 + B12×C14) + B12×B34×(C41×B23 + B41×C23) ;
RR24 ← N24×B41×B12×B33 ;
%
%
RRR10 ← N14×N32×A21×A43 ;
RRR15 ← N31×N14×A43×A22 ;
RRR17 ← N31×N13×A22×A44 ;
RRR19 ← B41×B23×B12×B34 ;
RRR12 ← N13×N32×A21×A44 ;
%
%
D15 ← RA1×B22 + RB1×A22 ;
F15 ← RA3×B21 + RB3×A21 ;
E15 ← RA2×B22 + RB2×A22 ;
G15 ← RA4×B21 + RB4×A21 ;
%
D14 ← RA1×C22 + RB1×B22 + RC1×A22 ;
E14 ← RA2×C22 + RB2×B22 + RC2×A22 ;
F14 ← RA3×C21 + RB3×B21 + RC3×A21 ;
```

```
G14 ← RA4×C21 + RB4×B21 + RC4×A21 ;
%
D13 ← RC1×B22 + RB1×C22 ;
E13 ← RC2×B22 + RB2×C22 ;
F13 ← RC3×B21 + RB3×C21 ;
G13 ← RC4×B21 + RB4×C21 ;
%
%
KA6 ←+RA1×A22×A44 ;
KB6 ←-RA2×A22×A43 ;
KC6 ←-RA3×A21×A44 ;
KD6 ← RA4×A21×A43 ;
%
KA5 ← A44×D15 + RA1×A22×B44 ;
KB5 ←-A43×E15 - RA2×A22×B43 ;
KC5 ←-A44×F15 - RA3×A21×B44 ;
KD5 ← A43×G15 + RA4×A21×B43 ;
KA4 ← A44×D14 + B44×D15 + RA1×A22×C44 ;
KB4 ←-A43×E14 - B43×E15 - RA2×A22×C43 ;
KC4 ←-A44×F14 - B44×F15 - C44×RA3×A21 ;
KD4 ← A43×G14 + B43×G15 + C43×RA4×A21 ;
KE4 ←-N32×RA8×A44 ;
KF4 ← RA13×A44 ;
KG4 ←-RA21×A22 ;
KH4 ← RA23×A22 ;
%
%
KA3 ← A44×D13 + B44×D14 + C44×D15 ;
KB3 ←-A43×E13 - B43×E14 - C43×E15 ;
KC3 ←-A44×F13 - B44×F14 - C44×F15 ;
KD3 ← A43×G13 + B43×G14 + C43×G15 ;
KE3 ←-N32×(RA8×B22 + RB8×A44) ;
KF3 ← RA13×B44 + RB13×A44 ;
KG3 ←-RA21×B22 - RB21×A22 ;
KH3 ← RA23×B22 + RB23×A22 ;
%
```

```
%
KA2 ← RC1×C22×A44 + B44×D13 + C44×D14 ;
KB2 ←-RC2×C22×A43 - B43×E13 - C43×E14 ;
KC2 ←-RC3×C21×A44 - B44×F13 - C44×F14 ;
KD2 ← RC4×C21×A43 + B43×G13 + C43×G14 ;
KE2 ← -N32×(RA8×C44 + RB8×B22 + RC8×A44) ;
KF2 ←  RA13×C44 + RB13×B44 + RC13×A44 ;
KG2 ← -RA21×C22 - RB21×B22 - RC21×A22 ;
KH2 ←  RA23×C22 + RB23×B22 + RC23×A22 ;
%
%
KA1 ←  RC1×C22×B44 + C44×D13 ;
KB1 ← -RC2×C22×B43 - C43×E13 ;
KC1 ← -RC3×C21×B44 - C44×F13 ;
KD1 ←  RC4×C21×B43 + C43×G13 ;
KE1 ← -N32×(RC8×B22 + RB8×C44) ;
KF1 ←  RC13×B44 + RB13×C44 ;
KG1 ← -RC21×B22 - RB21×C22 ;
KH1 ←  RC23×B22 + RB23×C22 ;
%
%
%
KA0 ←  RC1×C22×C44 ;
KB0 ← -RC2×C22×C43 ;
KC0 ← -RC3×C21×C44 ;
KD0 ←  RC4×C21×C43 ;
KE0 ← -N32×RC8×C44 ;
KF0 ←  RC13×C44    ;
KG0 ← -RC21×C22    ;
KH0 ←  RC23×C22    ;
FF ← KA6 + KB6 + KC6 + KD6 ;
GG ← KA5 + KB5 + KC5 + KD5 ;
HH ← KA4 + KB4 + KC4 + KD4 + KE4 + KF4 + KG4 + KH4 - RRR10 + RRR12 +
      RRR15 - RRR17 - RRR19 + XX5×RR5 + XX4×RR6 -RRR7 - XX5×RR9 +
   XX5×RR11 - XX4×RR18 + XX4×RR24 ; % ADD TO HH
JJ ← KA3 + KB3 + KC3 + KD3 + KE3 + KF3 + KG3 + KH3 + RR5 + RR6 - RR9
```

```
     -RR10 + RR11 + RR12 + RR15 - RR17 - RR18 - RR19 + RR24 + XX4×RA6 +
   XX5×RA5 - RR7 - XX5×RA9 + XX5×RA11 - XX4×RA18 + XX4×RA24 $%ADD TO JJ
KK ← KA2 + KB2 + KC2 + KD2 + KE2 + KF2 + KG2 + KH2 + RA5 + RA6
     -RA9 - RA10 + RA11 + RA12 + RA15 - RA17 - RA18 - RA19 + RA22 +RA24
 + XX5×RB5 + XX4×RB6 - RA7 - XX5×RB9 + XX5×RB11 - XX5×RB16
 - XX4×RB18 - XX4×RB20 + XX4×RB24 + N31×N13×XX0 $%ADD TO KK
LL ← KA1 + KB1 + KC1 + KD1 + KE1 + KF1 + KG1 + KH1 + RB5 + RB6
     - RB9 - RB10 + RB11 + RB12 + RB15 - RB16 - RB17 - RB18 - RB19
 + XX5×RC5 + XX4×RC6 - RB7 - XX5×RC9 + XX5×RC11 + N31×N13×XX1 -
   XX5×RC16 - XX4×RC18 - XX4×RC20 + XX4×RC24 -RB20 + RB22 +RB24$%ADD TO LL
NN ← KA0 + KB0 + KC0 + KD0 + KE0 + KF0 + KG0 + KH0 + RC5 + RC6
     - RC9 - RC10 + RC11 + RC12 +        RC15 - RC16 - RC17 - RC18
     - RC19 - RC20 + RC22 + RC24 - RC7 + N31×N13×XX2          $
%
%
%
%  PUT THESE VALUES BEFORE THE  DZ
TA3 ← B11×B33×BM5 $
TB3 ← B11×B33×BM3 $
TC3 ← B11×B34×BM5 $
TD3 ← B11×B34×BM3 $
TE3 ← BM2×B12×B33 $
TF3 ← BM5×B12×B33 $
TG3 ← BM2×B34×B12 $
TH3 ← B34×BM5×B12 $
%
%
TA2 ← BM5×(C11×B33 + B11×C33) + B11×B33×CM5 $
TB2 ← BM3×(C11×B33 + B11×C33) + B11×B33×CM3 $
TC2 ← BM5×(C11×B34 + B11×C34) + B11×B34×CM5 $
TD2 ← BM3×(C11×B34 + B11×C34) + B11×B34×CM3 $
TE2 ← B33×(CM2×B12 + BM2×C12) + BM2×B12×C33 $
TF2 ← B33×(CM5×B12 + BM5×C12) + BM5×B12×C33 $
TG2 ← B12×(CM2×B34 + BM2×C34) + BM2×B34×C12 $
TH2 ← B12×(C34×BM5 + B34×CM5) + B34×BM5×C12 $
%
```

```
%
TA1 ← CM5×(C11×B33 + B11×C33) + C11×C33×BM5 ;
TB1 ← CM3×(C11×B33 + B11×C33) + C11×C33×BM3 ;
TC1 ← CM5×(C11×B34 + B11×C34) + C11×C34×BM5 ;
TD1 ← CM3×(C11×B34 + B11×C34) + C11×C34×BM3 ;
TE1 ← C33×(CM2×B12 + BM2×C12) + CM2×C12×B33 ;
TF1 ← C33×(CM5×B12 + BM5×C12) + CM5×C12×B33 ;
TG1 ← C12×(CM2×B34 + BM2×C34) + CM2×C34×B12 ;
TH1 ← C12×(C34×BM5 + B34×CM5) + C34×CM5×B12 ;
%
%
TA0 ← C11×C33×CM5 ;
TB0 ← C11×C33×CM3 ;
TC0 ← C11×C34×CM5 ;
TD0 ← C11×C34×CM3 ;
TE0 ← CM2×C12×C33 ;
TF0 ← CM5×C12×C33 ;
TG0 ← CM2×C34×C12 ;
TH0 ← C34×CM5×C12 ;
%
BEGIN REAL IMAGBUCKLE ;
DZ1  ←  N24×N32×B11×BM5 ;
DZ2  ← −N32×(BM5×(C11×B23 + B11×C23) + B11×B23×CM5) ;
DZ3  ← −N14×N32×(A21×CM5 + B21×BM5) ;
DZ4  ←  N13×N32×(A21×CM5 + B21×BM5) ;
DZ5  ← −N13×N42×BM2×B34 ;
DZ6  ← −N14×N32×(A43×CM2 + B43×BM2) ;
DZ7  ←  N14×N42×BM2×B33 ;
DZ8  ←  N13×N32×(A44×CM2 + B44×BM2) ;
DZ9  ←  N31×(B12×(C23×BM5 + B23×CM5) + B23×BM5×C12);
DZ10 ←  N14×N31×(A22×CM5 + B22×BM5) ;
DZ11 ←  N31×N14×(CM3×A43 + BM3×B43) ;
DZ12 ← −N31×N13×(A44×CM3 + BM3×B44) ;
DZ13 ← −N13×N31×(BM5×B22 + CM5×A22) ;
DZ14 ← −N31×N24×BM5×B12 ;
DZ15 ← −N14×(B33×(C41×BM3 + B41×CM3) + B41×BM3×C33) ;
```

```
DZ16 ←  N13×(B34×(C41×BM3 + B41×CM3) + B41×BM3×C34) ;
DZ17 ←  TA0×A22 + TA1×B22 + TA2×C22 ;
DZ18 ←  TB0×A44 + TB1×B44 + TB2×C44 ;
DZ19 ← -TC0×A22 - TC1×B22 - TC2×C22 ;
DZ20 ← -TD0×A43 - TD1×B43 - TD2×C43 ;
DZ21 ← -TE0×A44 - TE1×B44 - TE2×C44 ;
DZ22 ← -TF0×A21 - TF1×B21 - TF2×C21 ;
DZ23 ←  TG0×A43 + TG1×B43 + TG2×C43 ;
DZ24 ←  TH0×A21 + TH1×B21 + TH2×C21 ;
%
%  SUBSTITUTE AFTER DZ
%
%
EZ1   ←  N24×N32×(B11×CM5 + C11×BM5) ;
EZ2   ← -N32×(CM5×(C11×B23 + B11×C23) + C11×C23×BM5) ;
EZ3   ← -N14×N32×(B21×CM5 + C21×BM5) ;
EZ4   ←  N13×N32×(B21×CM5 + C21×BM5) ;
EZ5   ← -N13×N42×(BM2×C34 + CM2×B34) ;
EZ6   ← -N14×N32×(B43×CM2 + C43×BM2) ;
EZ7   ←  N14×N42×(B33×CM2 + C33×BM2) ;
EZ8   ←  N13×N32×(B44×CM2 + BM2×C44) ;
EZ9   ←  N31×(C12×(C23×BM5 + B23×CM5) + C23×CM5×B12) ;
EZ10 ←  N14×N31×(B22×CM5 + BM5×C22) ;
EZ11 ←  N31×N14×(B43×CM3 + BM3×C43) ;
EZ12 ← -N31×N13×(CM3×B44 + C44×BM3) ;
EZ13 ← -N31×N13×(BM5×C22 + B22×CM5) ;
EZ14 ← -N31×N24×(CM5×B12 + BM5×C12) ;
EZ15 ← -N14×(C33×(C41×BM3 + B41×CM3) + C41×CM3×B33) ;
EZ16 ←  N13×(C34×(C41×BM3 + B41×CM3) + C41×CM3×B34) ;
EZ17 ←  TA0×B22 + TA1×C22 ;
EZ18 ←  TB0×B44 + TB1×C44 ;
EZ19 ← -TC0×B22 - TC1×C22 ;
EZ20 ← -TD0×B43 - TD1×C43 ;
EZ21 ← -TE0×B44 - TE1×C44 ;
EZ22 ← -TF0×B21 - TF1×C21 ;
EZ23 ←  TG0×B43 + TG1×C43 ;
```

```
EZ24 ←  TH0×B21 + TH1×C21 ;
%
%
FZ1  ←  N24×N32×C11×CM5 ;
FZ2  ← -C11×C23×CM5×N32 ;
FZ3  ← -N14×N32×C21×CM5 ;
FZ4  ←  N13×N32×C21×CM5 ;
FZ5  ← -N13×N42×CM2×C34 ;
FZ6  ← -N14×N32×CM2×C43 ;
FZ7  ←  N14×N42×CM2×C33 ;
FZ8  ←  N13×N32×CM2×C44 ;
FZ9  ←  C23×CM5×C12×N31 ;
FZ10 ←  N31×CM5×C22×N14 ;
FZ11 ←  N31×N14×CM3×C43 ;
FZ12 ← -N31×N13×CM3×C44 ;
FZ13 ← -N31×N13×CM5×C22 ;
FZ14 ← -N31×N24×CM5×C12 ;
FZ15 ← -N14×C41×CM3×C33 ;
FZ16 ←  N13×C41×CM3×C34 ;
FZ17 ←  TA0×C22 ;
FZ18 ←  TB0×C44 ;
FZ19 ← -TC0×C22 ;
FZ20 ← -TD0×C43 ;
FZ21 ← -TE0×C44 ;
FZ22 ← -TF0×C21 ;
FZ23 ←  TG0×C43 ;
FZ24 ←  TH0×C21 ;
%
%
CZ2  ← -N32×B11×B23×BM5 ;
CZ3  ← -N14×N32×A21×BM5 ;
CZ4  ←  N13×N32×A21×BM5 ;
CZ6  ← -N14×N32×BM2×A43 ;
CZ8  ←  N13×N32×A44×BM2 ;
CZ9  ←  N31×B23×BM5×B12 ;
CZ10 ←  N31×N14×BM5×A22 ;
```

```
CZ11 ←  N31×N14×BM3×A43 ;
CZ12 ← -N31×N13×BM3×A44 ;
CZ13 ← -N31×N13×BM5×A22 ;
CZ15 ← -N14×B41×BM3×B33 ;
CZ16 ←  N13×B41×BM3×B34 ;
CZ17 ←  TA1×A22 + TA2×B22 + TA3×C22 ;
CZ18 ←  TB1×A44 + TB2×B44 + TB3×C44 ;
CZ19 ← -TC1×A22 - TC2×B22 - TC3×C22 ;
CZ20 ← -TD1×A43 - TD2×B43 - TD3×C43 ;
CZ21 ← -TE1×A44 - TE2×B44 - TE3×C44 ;
CZ22 ← -TF1×A21 - TF2×B21 - TF3×C21 ;
CZ23 ←  TG1×A43 + TG2×B43 + TG3×C43 ;
CZ24 ←  TH1×A21 + TH2×B21 + TH3×C21 ;
%
%
BZ17 ←  TA2×A22 + TA3×B22 ;
BZ18 ←  TB2×A44 + TB3×B44 ;
BZ19 ← -TC2×A22 - TC3×B22 ;
BZ20 ← -TD2×A43 - TD3×B43 ;
BZ21 ← -TE2×A44 - TE3×B44 ;
BZ22 ← -TF2×A21 - TF3×B21 ;
BZ23 ←  TG2×A43 + TG3×B43 ;
BZ24 ←  TH2×A21 + TH3×B21 ;
%
%
AZ17 ←  TA3×A22 ;
AZ18 ←  TB3×A44 ;
AZ19 ← -TC3×A22 ;
AZ20 ← -TD3×A43 ;
AZ21 ← -TE3×A44 ;
AZ22 ← -TF3×A21 ;
AZ23 ←  TG3×A43 ;
AZ24 ←  TH3×A21 ;
%
%
PP ← AZ17 + AZ18 + AZ19 + AZ20 + AZ21 + AZ22 + AZ23 + AZ24 ;
```

```
QQ ← BZ17 + BZ18 + BZ19 + BZ20 + BZ21 + BZ22 + BZ23 + BZ24 ;
RR ← CZ2 + CZ3 + CZ4 + CZ6 + CZ8 + CZ9 + CZ10 + CZ11 + CZ12 + CZ13 +
     CZ15 + CZ16 + CZ17 + CZ18 + CZ19 + CZ20 + CZ21 + CZ22+CZ23 + CZ24
   + XX4×DZ1 + XX5×DZ5 + XX5×DZ7 + XX4×DZ14 ; % ADD TO RR
SS ← DZ1 + DZ2 + DZ3 + DZ4 + DZ5 + DZ6 + DZ7 + DZ8 + DZ9 + DZ10 + DZ11
   + DZ12 + DZ13 + DZ14 + DZ15 + DZ16 + DZ17 + DZ18 + DZ19 + DZ20 +
     DZ21 + DZ22 + DZ23 + DZ24
   + XX4×EZ1 + XX5×EZ5 + XX5×EZ7 + XX4×EZ14 ; % ADD TO SS
TT ← EZ1 + EZ2 + EZ3 + EZ4 + EZ5 + EZ6 + EZ7 + EZ8 + EZ9 + EZ10 + EZ11
   + EZ12 + EZ13 + EZ14 + EZ15 + EZ16 + EZ17 + EZ18 + EZ19 + EZ20 +
     EZ21 + EZ22 + EZ23 + EZ24
   + XX4×FZ1 + XX5×FZ5 + XX5×FZ7 + XX4×FZ14 ; % ADD TO TT
VV ← FZ1 + FZ2 + FZ3 + FZ4 + FZ5 + FZ6 + FZ7 + FZ8 + FZ9 + FZ10 + FZ11
   + FZ12 + FZ13 + FZ14 + FZ15 + FZ16 + FZ17 + FZ18 + FZ19 + FZ20 +
     FZ21 + FZ22 + FZ23 + FZ24 ;
EE ← VV ← NN ← 0 ;
% Q2 ← AA×LA*4 + BB×LA*3 + CC×LA*2 + DD×LA + EE
% Q1 ← PP×LA*5 + QQ×LA*4 + RR×LA*3 + SS×LA*2 + TT×LA + VV
% Q0 ← FF×LA*6 + GG×LA*5 + HH×LA*4 + JJ×LA*3 + KK×LA*2 + LL×LA + NN
%
%
RC[8] ← FF × PP ;
RC[7] ← 2 × FF × QQ ;
RC[6] ← 3 × FF × RR + GG × QQ - PP × HH ;
RC[5] ← 4 × FF × SS + 2 × GG × RR - 2 × PP × JJ ;
RC[4] ← 5 × FF × TT + 3 × GG × SS + HH × RR - JJ × QQ - 3 × PP × KK ;
RC[3] ← 4 × GG × TT + 2 × HH × SS - 2 × KK × QQ - 4 × LL × PP ;
RC[2] ← 3 × HH × TT + JJ × SS - RR × KK - 3 × QQ × LL ;
RC[1] ← 2 × JJ × TT - 2 × RR × LL ;
RC[0] ← KK × TT - SS × LL ;
FOR I ← 4 STEP -1 UNTIL 1 DO
RC[I] ← RC[I]/RC[0] ;
RC[0] ← 1 ;
BEGIN REAL SPHERE ;
         FOR N←0 STEP 1 UNTIL 8  DO
    IC[N] ← 0 ;
```

```
          SOLVE( 4,3,R ,J ,PRINT,FUNCT) ;
     FOR I ← 1 STEP 1 UNTIL 4  DO
BEGIN
          IF J[I] = 0
          THEN BEGIN
Q1[I] ← PP×R[I]*5 + QQ×R[I]*4 + RR×R[I]*3 + SS×R[I]*2 + TT×R[I] + VV ;
Q0[I] ← FF×R[I]*6 + GG×R[I]*5 + HH×R[I]*4 + JJ×R[I]*3 + KK×R[I]*2 +
        LL×R[I] + NN ;
PGLOBAL[I] ← - Q0[I]/Q1[I] ;
          WRITE(PRINT,RESULTS,LISTRESULTS) ;
%
END
          ELSE
BEGIN
WRITE(PRINT[NO],IMAGFMT,IMAGLIST) ;
WRITE(PRINT,FOMMAT) ;
          END ;
END ;
END ;
END ;
END ;
END ;
GO TO READCARD ;
FINISH : END.
```

VITA

John Palmer Anderson was born in New Orleans, Louisiana on March 27, 1939, one of identical twin brothers. He subsequently attended elementary and grade schools in New Orleans, Sarasota, Florida and Galveston, Texas. He attended Glynn Academy high school in Brunswick, Georgia, and graduated in 1956. John enrolled as an electrical engineering student on the co-op plan at the Georgia Institute of Technology in September, 1956. Upon completion of his four years of alternate work quarters at Hercules Powder Company in Brunswick and school quarters, he transferred to the School of Mathematics at Georgia Tech. John received his B.S. in Applied Mathematics (with honor) in June, 1961. He then enrolled for graduate study at Georgia Tech in both the School of Applied Mathematics and the School of Engineering Mechanics. He received the M.S. in Engineering Mechanics in June, 1963 and his M.S. in Applied Mathematics in June, 1964. He then enrolled as a doctoral student in engineering mechanics.

On June 16, 1962, John married Mary Agnes Harris of Griffin, Georgia. They have a daughter, Deborah, born on March 30, 1965.

During the years 1961 to 1964, John worked as a graduate assistant in the School of Applied Mathematics and the School of Engineering Mechanics, teaching mathematics and mechanics. From 1964 to 1965, he held a NASA fellowship, and during the school year 1965-1966, he held the position of Assistant Professor in the School of Engineering Mechanics. Upon completion of his doctorate, John will depart for the United States

Air Force Academy in Colorado to serve his two year military obligation teaching in the mechanics department.